ISOMORPHISMS IN SUBSPACES OF $l_1$

ROBERT H. LOHMAN

Abstract. It is shown that a closed, infinite dimensional linear subspace $X$ of $l_1$ is quotient homogeneous if and only if $X$ is isomorphic to $l_1$.

In a recent paper [1] Lindenstrauss and Rosenthal defined two types of homogeneity for Banach spaces. A Banach space $X$ is said to be subspace homogeneous if for every two isomorphic closed subspaces $Y$ and $Z$ of $X$, both of infinite codimension, there is an automorphism (i.e. a bounded linear bijection of $X$) which carries $Y$ onto $Z$. $X$ is said to be quotient homogeneous if whenever $Y$ and $Z$ are two closed, infinite dimensional subspaces of $X$ with $X/Y$ isomorphic to $X/Z$, then there is an automorphism of $X$ which carries $Y$ onto $Z$. It was shown in [1] that $c_0$ is subspace homogeneous and that $l_1$ is quotient homogeneous. In [2] Lohman showed that a closed, infinite dimensional subspace $X$ of $c_0$ is subspace homogeneous if and only if $X$ is isomorphic to $c_0$. The purpose of this note is to show that every infinite dimensional quotient homogeneous subspace of $l_1$ is isomorphic to $l_1$. We follow the notation of [1].

Theorem. Let $X$ be a closed, infinite dimensional subspace of $l_1$. $X$ is quotient homogeneous if and only if $X$ is isomorphic to $l_1$.

Proof. If $X \cong l_1$, then $X$ is quotient homogeneous by the results of [1].

On the other hand, assume $X$ is quotient homogeneous. $X$ contains a subspace $Y$, complemented in $l_1$, such that $Y \cong l_1$ [3, Lemma 2]. We may write $X = Y \oplus Z$ for a closed subspace $Z$ of $X$. If $Z$ is finite dimensional, then $X \cong l_1$. Hence we may assume $Z$ is infinite dimensional. Recall that every separable Banach space is isomorphic to a quotient of $l_1$. It follows that there exists a closed subspace $Y_1$ of $Y$ such that $Y/Y_1 \cong Z$. Now $Y_1 \oplus Z$ is closed and

$$X/(Y_1 \oplus Z) \cong Y/Y_1 \cong Z \cong X/Y.$$
Since $X$ is quotient homogeneous, it follows that

$$ Y_1 \oplus Z \approx Y \approx l_1. $$

As an infinite dimensional, complemented subspace of an isomorphic copy of $l_1$, we have $Z \approx l_1$ [3, Theorem 1]. Therefore $X \approx l_1$, completing the proof.

REFERENCES


Department of Mathematics, Kent State University, Kent, Ohio 44240