NONASYMPTOTICALLY ABELIAN FACTORS OF TYPE III

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ABSTRACT. There exists a continuum of nonisomorphic non-asymptotically abelian type III factors on separable Hilbert spaces.

1. A von Neumann algebra $R$ is called \textit{asymptotically abelian} if there exists a sequence $\{x_n\}$ of $\ast$-automorphisms of $R$ such that for any $S, T \in R$, $[x_n(S), T] \to 0$ strongly, where $[A, B] = AB - BA$, i.e. $\{x_n(S)\}$ is a central sequence\textsuperscript{2} for each $S \in R$. This definition was introduced by Sakai in [6], where it was proved that no finite type I factor is asymptotically abelian, and that there exist asymptotically abelian and nonasymptotically abelian factors of type $\mathrm{II}_1$, both having nontrivial central sequences. In [7] Willig proved that the type $\mathrm{I}_\infty$ factor is not asymptotically abelian. Glaser [4] showed that no type $\mathrm{II}_1$ factor is asymptotically abelian, and also applied the method of [7] to show that one example of type III factors in Pukanszky [5] is not asymptotically abelian. Indeed, this particular factor has no nontrivial central sequence. In this note we shall show that the continuum of nonisomorphic type III factors constructed in [2] consists entirely of nonasymptotically abelian factors each with nontrivial central sequences.

2. Let $R$ be a von Neumann algebra. A von Neumann subalgebra $B$ of $R$ is called a \textit{residual subalgebra} of $R$ if for any central sequence $\{T_n\}$ in $R$ there exists a bounded sequence $\{S_n\}$ in $B$ such that $T_n - S_n \to 0$ strongly.

THEOREM. Let $R$ be a properly infinite von Neumann algebra on a Hilbert space $H$. Let $x \in H$ be a unit vector, and $B$ a residual subalgebra of $R$ be such that $(STx|x) = (TSx|x)$ for all $T \in R$ and $S \in B$. Then $R$ is not asymptotically abelian.

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\textsuperscript{2} A bounded sequence $\{T_n\}$ of operators in a von Neumann algebra $R$ is called \textit{central} if $[T_n, T] \to 0$ strongly for all $T \in R$. The central sequence $\{T_n\}$ is called \textit{trivial} if there exists a sequence $\{c_n\}$ of complex numbers such that $T_n - c_n I \to 0$ strongly. See Dixmier and Lance [3].

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Proof. Suppose that \( R \) is asymptotically abelian, and that \( \{\alpha_n\} \) is the requisite sequence of \(*\)-automorphisms of \( R \). Define a sequence of functionals on \( R \) by

\[
\phi_n(T) = (\alpha_n(T)x | x), \quad T \in R.
\]

Since \( \|\phi_n\| = 1 \), by the weak* compactness of the unit ball of the dual of \( R \), there exists a subsequence of \( \{\phi_n\} \) (which we again call \( \{\phi_n\} \)) converging weakly to some \( \phi_0 \) in the unit ball. Now for any \( T, S \in R \), \( \{\alpha_n(T)\} \) and \( \{\alpha_n(S)\} \) are central sequences, and hence there exist sequences \( \{T_n\} \) and \( \{S_n\} \) in \( B \) such that \( \alpha_n(T) - T_n \to 0 \) strongly and \( \alpha_n(S) - S_n \to 0 \) strongly.

Since strong convergence of operators is jointly continuous on bounded sets, it follows that

\[
\alpha_n(T)\alpha_n(S) - T_n\alpha_n(S) + T_nS_n \to 0 \quad \text{strongly}.
\]

Now note that

\[
\phi_0(TS) = \lim \langle \alpha_n(T)\alpha_n(S)x | x \rangle = \lim \langle \alpha_n(T)S_n + T_n\alpha_n(S) - T_nS_n | x \rangle
\]

\[
= \lim \langle S_n\alpha_n(T) + \alpha_n(S)T_n - S_nT_n | x \rangle = \phi_0(ST).
\]

Thus \( \phi_0 \) is a finite trace on \( R \), contradicting the assumption that \( R \) is properly infinite. Q.E.D.

Corollary. There exists a continuum of nonisomorphic nonasymptotically abelian factors of type III.

Proof. Let \( R_x = M(X, \mu) \otimes \Delta \) be the factor of type III on the Hilbert space \( H_1 = L^2(X, \mu) \otimes L^1(\Delta) \) with \( 1 \otimes \delta_\varepsilon \) as a separating cyclic vector, as constructed in \$3\$ of [1] (or see [5]). Let \( \{G_\alpha\}_{\alpha \in I} \) be the continuum of groups constructed in \$2\$ of [2]. Let \( R_x = R_1 \otimes \phi G_\alpha \) be the factor of type III on the Hilbert space \( H = H_1 \otimes L^1(G_\alpha) \) with a separating cyclic unit vector \( \xi = 1 \otimes \delta_\varepsilon \otimes \delta_\varepsilon \). By Remark 4 of [2], \( B_x = M(X, \mu) \otimes \mathcal{A}(H_N) \) is residual in \( R_x \).

As observed in the proof of that remark, we have

\[
(TS\xi | \xi) = (ST\xi | \xi) \quad \text{for any } T \in R_x, S \in B_x.
\]

Hence by the theorem above and by Theorem 1 in [2], \( \{R_x\}_{x \in I} \) is a continuum of pairwise nonisomorphic nonasymptotically abelian factors of type III. Q.E.D.

Remark 1. If \( R \) is properly infinite and has only trivial central sequences, then take \( B = C \), the field of complex numbers, and \( x \) any unit vector in the above Theorem, so that \( R \) is not asymptotically abelian. Thus any asymptotically abelian properly infinite von Neumann algebra must have nontrivial central sequence. It is worth noting, in this regard, that the factors \( \{R_x\}_{x \in I} \) all have nontrivial central sequences.
Remark 2. By the same proof as in the Corollary, the factors $R_3$ and $R_4$ constructed in [1] are not asymptotically abelian.

Finally, we remark that it is not known whether the corresponding $\text{II}_1$-factors $\{\mathcal{A}(G_a)\}_{a \in A}$ constructed in [2] are asymptotically abelian or not.

Bibliography


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