FIBERED KNOTS THROUGH T-SURGERY

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Abstract. Noninvertible fibered knots are constructed by a discrete surgical procedure called T-surgery on links.

A particular surgical technique will transform certain two-component links into knots. The process is called surgical transmutation or, more simply, T-surgery; the links used for this surgery are called admissible, and a knot obtained from an admissible link $L$ is a transmute of $L$. The immediate importance of T-surgery is that one can produce knots with desired properties by its application to appropriately chosen admissible links; for example, the method yields a veritable cornucopia of noninvertible knots (\[11\], \[12\]).

The basic objective of this paper is to prove the existence of noninvertible fibered knots by T-surgery. These knots stand in contrast to H. F. Trotter's noninvertible, nonfibered pretzel knots \[9\], and to such knots as 8\textsubscript{17} that are easily seen to be fibered (by the Murasugi test \[4\], Theorem 1.2, p. 544], say) but whose intuitively obvious noninvertibility feature has not yet been proved \[2\], Problem 10, p. 169].

In §1, we describe our surgical method, and give necessary and sufficient conditions (Theorem 1) for a transmute to be a prime knot. Our principal result (Theorem 2 of §2) states geometric conditions for a transmute to fiber. Intuitively Theorem 2 and the examples of §3 indicate an abundant variety of noninvertible fibered knots.

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1. T-surgery and prime transmutes. An oriented, ordered link $L = K_1 \cup K_2$ of two components tamely imbedded in the oriented three-sphere $\mathbb{S}$ is admissible if $K_1$ is of trivial knot type and $L$ is unsplittable. Let $L$ be an admissible link, denote by $W$ the closure of the complement in $\mathbb{S}$ of a tubular neighborhood of $K_1$ in $\mathbb{S} - K_2$, and let $V$ denote a knotted,
closed solid torus, namely imbedded in $\mathcal{S}$. The (oriented) image $\mathcal{N}$ of $K_2$ under a faithful homeomorphism [7] of $W$ onto $V$ is a *transmute of $L$ with respect to $V*$, and we shall say that $\mathcal{N}$ was obtained from $L$ by *surgical transmutation* (or *T-surgery*).

**Theorem 1.** Let $L=K_1 \cup K_2$ be an admissible link, and let $\mathcal{N}$ denote a knot obtained by T-surgery on $L$. In order that $\mathcal{N}$ be a prime knot, it is both necessary and sufficient that $L$ be a prime link and that the order [7] of $W$ with respect to $K_2$ be $\geq 2$.

A proof of Theorem 1 may be constructed by straightforward applications of the definition of T-surgery and several theorems of H. Schubert [7]; we shall omit the details. Evidently, the transmutes of the link of Figure 1 in each of [11], [12], and the present paper are prime knots.

![Figure 1](image)

Theorem 1 can, with little trouble, also be applied to the links $\mathcal{L}_{(+)}$ and $\mathcal{L}_{(-)}$ (Figures 4 and 6, respectively of [12]).

2. **Fibered transmutes.** One should perhaps consider an example (Figure 1 of [11], [12], or the present paper) while reading

**Theorem 2.** $L=K_1 \cup K_2$ is an admissible link, and $V_1$ and $V_2$ represent disjoint, tubular neighborhoods of $K_1$ and $K_2$, respectively. Set $W=\mathcal{S}-\text{Int } V_1$, let $V$ denote a knotted, tame, closed solid torus in $\mathcal{S}$, and suppose $K$ is an (oriented) core of $V$. We assume that $L$ has the following additional properties:

(a) $K_2$ is a fibered knot;
(b) the order of $W$ with respect to $K_2$ coincides with the winding number $w$ (see [7]) of $K_2$ in $W$;

(c) there is an orientable surface $S^*$ of minimal genus properly imbedded in $\mathcal{S} - \text{Int } V_2$ and spanning a longitude of $V_2$, whose intersection with $\partial W$ consists of exactly $w$ longitudes of $W$.

Denote by $M^3$ the space $W - \text{Int } V_2$ split along the surface $S = S^* - \text{Int } V_1$ (see [5]). Besides the surfaces $S_1$ and $S_2$ that identify to yield $S$, and an annulus $\partial$ that closes to become $\partial V_2$, there are $w$ (closed) cylinders $\mathcal{C}_1, \ldots, \mathcal{C}_w$ such that

$$\partial M^3 = S_1 \cup S_2 \cup \partial \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_w.$$ 

Finally, suppose the following two conditions are satisfied:

(d) $K$ is a fibered knot;

(e) there exist $w$ pairwise disjoint, polyhedral disks $\mathcal{D}_1, \ldots, \mathcal{D}_w$ properly imbedded in $M^3$ such that $\partial \mathcal{D}_i = \alpha_i \cup \beta_i$, where $\alpha_i$ is a properly imbedded arc of $\mathcal{C}_i$ connecting the two components of $\partial \mathcal{D}_i$, and where $\beta_i$ is an arc properly imbedded in $S_1 \cup S_2 \cup \partial$ ($i = 1, \ldots, w$).

Our conclusion is that the transmute $\mathcal{X}$ of $L$ with respect to $V$ is a fibered knot.

**Proof.** According to [7, Theorem, p. 192], our hypothesis (c) implies that the genus of $\mathcal{X}$ is $Wg(K) + g(K_2)$, where $g(k)$ is the genus of a knot $k$. Let $f$ be a faithful homeomorphism of $W$ onto $V$. An orientable surface $\bar{g}$ of minimal genus properly imbedded in $\mathcal{S} - \text{Int } fV_2$ and spanning a longitude of $\partial (fV_2)$ may be constructed as follows: the orientable surface $fS$ meets $\partial V$ in $w$ (pairwise disjoint) longitudes $\lambda_1, \ldots, \lambda_w$. Let $\bar{g}_1, \ldots, \bar{g}_w$ denote orientable surfaces, each of genus $g(K)$ and properly imbedded in $\mathcal{S} - \text{Int } V$, such that $\bar{g}_i$ spans $\lambda_i$ ($i = 1, \ldots, w$), and $\bar{g}_i \cap \bar{g}_j = \emptyset$ $(i \neq j)$. Evidently, $\bar{g} = \bar{g}_1 \cup \cdots \cup \bar{g}_w \cup fS$ is a surface of the type desired.

We now take a countable number of separate copies of $\mathcal{S} - \text{Int } (fV_2)$ split along $\bar{g}$, and paste them together to obtain the infinite cyclic covering space $\Sigma_X$ of $\mathcal{S} - \text{Int } V$ [5, p. 27]. Clearly, $\Sigma_X$ is the union of $w+1$ spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_w, \mathcal{Y}_L$ such that each $\mathcal{Y}_i$ ($i = 1, \ldots, w$) is homeomorphic to the infinite cyclic covering space of $\mathcal{S} - \text{Int } V$, $\mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset$ $(i \neq j)$, and $\mathcal{Y}_L$ is homeomorphic to the covering space of $W - \text{Int } V_2$ obtained by properly pasting together a countable number of copies of $M^3$ (see hypothesis). We shall prove that $\pi_1(\Sigma_X)$ is free, whence the conclusion of Theorem 2 follows.

To begin with, $\Sigma_X = \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_w \cup \mathcal{Y}_L$, and $\mathcal{Y}_i \cap \mathcal{Y}_L = \mathcal{C}_i$ ($i = 1, \ldots, w$) is a covering space of $\partial V$ and homeomorphic to $S^1 \times \mathbb{R}$. The injections $\pi_1(\partial V) \rightarrow \pi_1(\mathcal{S} - \text{Int } V)$ and $\pi_1(\partial V) \rightarrow \pi_1(V - \text{Int } (fV_2))$ are monomorphisms; the first because $V$ is knotted [6], the second because the order $w$ of $W$ with
respect to $K_2$ is greater than zero. Consequently each of the injections \( \pi_1(\mathcal{C}_i) \to \pi_1(Y_i) \) and \( \pi_1(\mathcal{C}_i) \to \pi_1(Y_{ij}) \) \((i = 1, \cdots, w)\) is a monomorphism.

By an application of van Kampen's theorem [3] we may, therefore, write

\[
\pi_1(\Sigma_x) = \cdots (((\pi(\mathcal{Y}_1) \pi_1(\mathcal{Y}_2)) \cdots \pi_1(\mathcal{Y}_w), \pi_1(\mathcal{Y}_w)) \cdots \pi_1(\mathcal{Y}_2) \pi_1(\mathcal{Y}_1)),
\]

where \( \pi_1(\mathcal{C}_i) \approx \mathbb{Z} \) \((i = 1, \cdots, w)\).

Let \( \rho : \Sigma_x \to \Sigma \to \text{Int}(fV_2) \) be the covering associated with the covering space \( \Sigma_x \). If the surface \( fS \) supports the group \( \rho_*(\pi_1(Y_i)) \) (that is, if \( \rho_*(\pi_1(Y_i)) = \pi_1(fS) \)), then evidently \( S \) supports \( \rho_*(\pi_1(\Sigma_x)) \) since each \( \pi_1(Y_i) \) \((i = 1, \cdots, w)\) is free (hypothesis (d)). Thus, to complete the proof of Theorem 2, it suffices to prove that \( M^3 \) (see hypothesis) is homeomorphic to \( S \times I \) \((I = [0, 1])\).

If \( M^3 \) is cut open along the disks \( D_1, \cdots, D_w \) (see hypothesis (e)), we obtain a compact 3-manifold \( M^3 \) both homeomorphic to and a deformation retract of the space \( \mathcal{M}^3 \) obtained by splitting \( \Sigma \to \text{Int}(fV_2) \) along the surface \( S^* \) (hypothesis (a)). If \( S^*_1 \) and \( S^*_2 \) are the surfaces on \( \partial \mathcal{M}^3 \) closing under identification to \( S^* \), then \( \partial \mathcal{M}^3 = S^*_1 \cup S^*_2 \cup A. \) (We can assume that \( S_1 \subset S^*_1 \) \((i = 1, 2)\).) Since \( K_2 \) is a fibered knot (hypothesis (a)), \( \mathcal{M}^3 \) (and hence, \( M^3 \)) is a handlebody of genus \( 2g(K_2) \); in fact, \( M^3 \) is homeomorphic to \( S^* \times I \).

Now \( \partial M^3 \cap \partial \mathcal{M}^3 \) is a compact, connected, orientable surface of genus \( 2g(K_2) \) with \( w \) boundary components. We choose a canonical set \( \Phi \) of \( 2g(K_2) \) simple closed curves on \( S^*_1 \cap \partial M^3 \cap \partial \mathcal{M}^3 \) meeting only in a common point \( p \) (the basepoint of \( \pi_1(S^*_1) \)). These curves (properly oriented) represent a system of free generators for \( \pi_1(S^*_1) \). Now choose a system of pairwise disjoint cutting disks in \( \mathcal{M}^3 \), exactly one disk for each curve of our canonical set \( \Phi \) of curves and of transverse intersection with it. Evidently, each map of the deformation retraction of \( M^3 \) onto \( M^3 \) can be looked upon as a homeomorphism of \( \mathcal{M}^3 \) onto a subspace of itself leaving \( \partial M^3 \cap \partial \mathcal{M}^3 \) pointwise fixed (one simply "pushes in" each of the \( w \) (open) disks in \( \partial M^3 - \partial \mathcal{M}^3 \)).

The deformation retraction of \( \mathcal{M}^3 \) onto \( M^3 \) leads to a complete system \( \mathcal{D} \) of cutting disks for \( M^3 \). It is now clear that \( M^3 \) is a handlebody of genus \( w + 2g(K_2) \) (cf. [8]). Hypothesis (b) implies that \( S_1 \cap \mathcal{E}_i \) \((i = 1, \cdots, w)\) is a simple closed curve \( \gamma_i \).

By orienting \( \gamma_i \) and running an (oriented) arc \( \delta_i \) from \( p \) (the common intersection of the curves in \( \Phi \)) to \( \gamma_i \), we obtain a collection of curves \( \gamma_1 \cup \delta_1, \cdots, \gamma_w \cup \delta_w \) on \( S_1 \) that, in union with \( \Phi \), represents a system of free generators for \( \pi_1(S_1) \). Furthermore, each of these curves possesses transverse intersection with exactly one of the \( w + 2g(K_2) \) cutting disks in \( \mathcal{D} \cup \{D_1, \cdots, D_w\} \) of \( M^3 \). This means that the injection homomorphism \( \pi_1(S_1) \to \pi_1(M^3) \) is an isomorphism (cf. [8], [13]). Similarly, the injection \( \pi_1(S_2) \to \pi_1(M^3) \) is an isomorphism. It now follows...
from E. M. Brown's product theorem [1, Theorem 3.1, p. 485] that \( M^3 \) is homeomorphic to \( S_1 \times I \) and, hence, to \( S \times I \). This completes the proof of Theorem 2.

3. Noninvertible fibered knots: examples. One can readily see that \( T \)-surgery on the link of Figure 1 of [11] (or [12]) will yield an infinity of distinct noninvertible fibered knots. Figure 2 of [11] (Figure 3 of [12]) gives an example.

In order to cogently exemplify the variety of noninvertible fibered knots produced by \( T \)-surgery, we present one final example. Actually, for each positive integer \( n \), we obtain a link \( L_n \) (Figure 1) possessing an infinite collection of prime, noninvertible fibered transmutes; one such transmute is shown in Figure 2. Note that no two links \( L_n \) and \( L_m \) are of the same type unless \( n=m \) (cf. [10]). It is not difficult to show that the classes of transmutes (one for each \( L_n \)) are mutually disjoint. We shall omit proofs of the property claims for this example; these proofs are similar to those of the example in the above paragraph (see [12, §3]).
References


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