NIL ALGEBRAS SATISFYING AN IDENTITY
OF DEGREE THREE
RAYMOND COUGHLIN

Abstract. Let $A$ be a nonassociative algebra over a field $F$ with a function $g: A \times A \times A \rightarrow F$ such that $(xy)z = g(x, y, z)x(yz)$ for all $x, y,$ and $z$ in $A$. Algebras satisfying this identity have been studied by Michael Rich and the author. It is shown here that a finite-dimensional nil power-associative algebra satisfying the above identity is nilpotent.

Let $A$ be an algebra over a field $F$ with a function $g:A \times A \times A \rightarrow F$ such that

$$ (xy)z = g(x, y, z)x(yz) $$

for all $x, y, z$ in $A$. Semisimple algebras satisfying (1) and

$$ x^2x = xx^2 $$

have been studied by Michael Rich and the author [2]. A power-associative algebra $A$ is an algebra $A$ such that for every $x$ in $A$ the subalgebra generated by $x$ in $A$ is associative. A nil power-associative algebra is an algebra in which for every $x$ there exists an integer $n(x)$ such that $x^{n(x)}=0$. An algebra is nilpotent if there exists an integer $n$ such that $A^n=0$. It is shown that a finite-dimensional nil power-associative algebra satisfying (1) is nilpotent. Throughout this paper we will assume that all algebras are finite dimensional and power-associative. We define the associator $(x, y, z) = (xy)z - x(yz)$. The linearization of (2) is

$$ (x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) + (z, y, x) + (y, x, z) = 0. $$

We define $R(x)$ and $L(x)$ to be the usual endomorphisms on $AxR(x)=zx$ and $zL(x)=xz$. We can then write (1) in the form

$$ zR(a)R(b) = azR(ab), $$

$$ zL(a)R(b) = \beta zR(b)L(a) $$
for all \( z, a, b \) in \( A \) and \( x = g(z, a, b), \beta = g(a, z, b) \in F \). Throughout we assume that the characteristic of \( F \neq 2 \).

We begin with

**Theorem 1.** If \( A \) is an algebra over \( F \) satisfying (1) with \( x^N = 0 \), then the ring \( \mathcal{R} \) generated by the right and left multiplications by powers of \( x \) is a nilpotent ring.

**Proof.** From (1) we can write

\[
(6) \quad zL(x^n) = axL(x)^n
\]

for any \( z \) in \( A \) and where \( a \in F \) depends upon \( z, x \) and \( n \). For any element \( S \) in \( \mathcal{R} \) and \( z \) in \( A \) we can use (5) to pass all the right multiplications to the left and then use (4) and (5) to write \( zS \) as a linear combination of terms of the form \( zS = azR(x^m)L(x)^r \) for some \( a \in F \). Now let \( T \in \mathcal{R} \) and consider \( T^k \) where \( k = N + 2^N(N+1) \). We can write \( zT \) as a linear combination of terms of the form

\[
(7) \quad azR(x^m)L(x)^r
\]

where \( r \geq 2^N(N+1) \). We can choose a nonzero expression of this form such that \( r \) is minimal. From (3) we have

\[
(8) \quad zL(x^2) = zL(x^2) + \beta_1 zR(x^2) + \beta_2 zR(x)L(x)
\]

where \( \beta_1, \beta_2 \in F \). Into (7) substitute (8) for the two leftmost \( L(x) \)'s. By our choice of \( r \), (7) is equal to

\[
(9) \quad azR(x^m)L(x^2)L(x)^{r-2}.
\]

Into (9) substitute (8) again for the two leftmost \( L(x) \)'s and, continuing this process, we get that (7) is equal to

\[
(10) \quad azR(x^m)L(x^2)^{r_1}I_1(x)
\]

when \( I_1(x) \) is the identity operator or \( L(x) \), depending upon whether \( r \) is even or odd. Note that (6) allows us to still utilize the choice of \( r \) and hence to conclude that \( r_1 \) is minimal. Replace \( x \) by \( x^2 \) in (8) and then use this same process to write (10) in the form \( azR(x^m)L(x^4)^{r_j}I_q \) where \( I_q \) is either the identity operator, \( L(x) \), \( L(x^2) \) or \( L(x^3)L(x) \). In the same manner, again noting that (6) and the choice of \( r \) imply that \( r_{j-1} \) is minimal, we can write (10) in the form \( azR(x^m)L(x^4)^{r_j}I_q \). By the choice of \( r \) we have \( n_j \geq N \) for large enough \( j \) and this \( T^k = 0 \).

We denote by \( M(A) \) the associative algebra generated by the right and left multiplications of \( A \). If \( B \) is any subalgebra of \( A \) then \( B^* \) is the subalgebra of \( M(A) \) generated by the right and left multiplications of \( B \).
that is, every element in $B^*$ is a linear combination of terms of the form $S_1S_2\cdots S_n$ where each $S_i$ is a right or left multiplication by an element in $B$.

**Theorem 2.** Any finite-dimensional nil algebra $A$ over $F$ satisfying (1) is nilpotent.

**Proof.** As in the proof that any alternative nil algebra of finite dimension is nilpotent [3, p. 30], we let $B$ be a subalgebra of $A$ which is maximal with respect to the property that $B^*$ is nilpotent. Since the subalgebra $\{0\}$ has the property and $A$ is finite dimensional, such a maximal $B$ exists. We assume that $B$ is a proper subalgebra of $A$ and so there exists an element $x$ not in $B$ such that

$$x B^* \subseteq B. \tag{11}$$

We let

$$C = B + F[x], \tag{12}$$

so that $C^* = (B + F[x])^*$. We will show that $C^*$ is a nil algebra. Let $T \in C^*$. From (5) we can systematically pass all the right multiplications to the left and from (4) and (11) we can assume that each right multiplication is in $B^*$. Note that in this new expression for $zT$ the number of factors from $B^*$ is preserved. Hence we can assume that $zT$ is of the form

$$az R(b_1) R(b_2) \cdots R(b_r) L(a_1) L(a_2) \cdots L(a_s)$$

where the $a_i \in B$ or $Fx$, $a \in F$ and $z \in A$.

Now from (3) we can replace any $L(x)L(b)$ for $b \in B$ by a linear combination of terms of the form $L(b)L(x)$, $R(xb)$, $R(bx)$, $L(bx)$, $L(x)R(b)$, $L(b)R(x)$, $R(x)R(b)$, $R(b)L(x)$. Utilizing (4), (5) and (11) allows us to replace $L(x)L(b)$ by a linear combination of terms of the form $L(b)L(x)$, $R(b_1)$, $L(b_2)$, $R(x)L(b_3)$, $R(b_4)L(x)$. As before, all the right multiplications can be passed to the left and each $R(x)$ is enveloped by an $R(b)$. Hence for any $T$ in $C^*$, $zT$ can be written as a linear combination of terms of the form

$$az R(b_1) R(b_2) \cdots R(b_r) L(b_{r+1}) \cdots L(b_s)L(x)^i$$

or of the form

$$az R(x^i) L(b_1) L(b_2) \cdots L(b_s)L(x)^i$$

where the number of factors from $B^*$ is constant. If $B^{*N}=0$ and $x^m=0$, we have $T^{2nm\tilde{m}}=0$, where $\tilde{m}$ is the index of nilpotency of the ring mentioned in Theorem 1, because we have either $n$ factors from $B^*$ on the left or an element from $\{L(x)\}^\tilde{m}=0$, from Theorem 1. Hence every element of the
finite-dimensional associative algebra $C^*$ is nilpotent, and so $C^*$ is nilpotent [1, p. 23]. But $B$ was maximal with respect to the property of having $B^*$ nilpotent, so we have a contradiction, implying that $B$ is not a proper subalgebra of $A$. Thus $A = B$ and $A^*$ is nilpotent. It follows [3, p. 18] that $A$ is nilpotent.

REFERENCES


DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122