ON CLOSED ADDITIVE SEMIGROUPS IN $E^n$

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Abstract. Let $C(A)$ be the closed additive semigroup generated by a set $A \subseteq E^n$. A simple necessary and sufficient condition on $A$ for $C(A)$ to be a group is derived. An example which arose in the theory of random walks and stimulated these purely geometrical considerations is discussed at the end.

For an arbitrary subset $A$ of the $n$-dimensional Euclidean space $E^n (n \geq 1)$ let $S(A)$ denote the smallest additive semigroup containing $A$, $A^-$ the closure of $A$ with respect to the Euclidean topology, and $C(A) := (S(A))^\circ$. Obviously, $C(A)$ is the smallest closed additive semigroup containing $A$, i.e., the closed additive semigroup generated by $A$.

Definition. $A \subseteq E^n$ is called omnilateral, if $A \neq \emptyset$ and for every hyperplane $H$ through the origin with $A \perp H$ there are points of $A$ in each of the two open half-spaces produced by $H$.

In other words, $A$ is omnilateral if $A \neq \emptyset$ and $u \in E^n, x \in A$ and $ux > 0$ imply the existence of an $x^* \in A$ with $ux^* < 0$. Some immediate consequences are listed in the following lemma.

Lemma. (1) $A$ is omnilateral if and only if $S(A)$ is omnilateral. (2) $A$ is omnilateral if and only if $A^-$ is omnilateral. (3) An additive group $A$ is always omnilateral. (4) The image of an omnilateral set $A \subseteq E^n$ under a linear transformation from $E^n$ to $E^m$ is an omnilateral set in $E^m$.

We say a set $A \subseteq E^n$ is genuinely $n$-dimensional if there is no hyperplane through the origin $0$ containing $A$. $A^e$ denotes the convex hull of $A$.

Theorem 1. A genuinely $n$-dimensional set $A$ is omnilateral if and only if $0$ is an inner point of $A^e$.

Proof. The if-part is trivial. Suppose, on the other hand, $0$ is an exterior or a boundary point of $A^e$. There would be a hyperplane through...
the origin such that one of the corresponding open half-spaces contains no point of \( A \) \([3, \text{p. 20}]\). Thus \( A \) is not both omnilateral and genuinely \( n \)-dimensional.

**Theorem 2.** \( C(A) \) is a group if and only if \( A \) is omnilateral.

**Proof.** The only-if-part follows from the Lemma, parts (1), (2) and (3).

To prove the if-part, one can assume that \( A \) is genuinely \( n \)-dimensional. Let \( 0 \neq x_0 \in C(A) \) and \( \epsilon > 0 \) be given. Since 0 is an inner point of \( A^\epsilon \) there exists a \( \lambda_0 > 0 \) with \( -\lambda_0 x_0 \in A^\epsilon \). Hence there exist a natural number \( k \) and \( k \) points \( x_1, \cdots, x_k \in A \) and reals \( \lambda_1 > 0, \cdots, \lambda_k > 0 \) with

\[-\lambda_0 x_0 = \lambda_1 x_1 + \cdots + \lambda_k x_k.\]

By a theorem on approximations of real numbers by rational numbers (cf. for example \([4, \text{p. 170}]\)) there exist rational numbers \( p_i/q \) (\( i = 0, \cdots, k \); \( p_i \) and \( q \) natural numbers) with

\[|p_i/q - \lambda_i| < (q \cdot q^{1/(k+1)})^{-1} \quad (i = 0, \cdots, k),\]

where the common denominator \( q \) can be chosen arbitrarily large. If

\[\frac{1}{q^{1/(k+1)}} < \frac{\epsilon}{(k + 1)\max(|x_0|, \cdots, |x_k|)},\]

it follows that

\[|(p_i - \lambda_i q)x_i| < \epsilon/(k + 1) \quad (i = 0, \cdots, k)\]

and thus

\[\left| \sum_{i=0}^{k} p_i x_i \right| = \left| \sum_{i=0}^{k} (p_i - \lambda_i q)x_i \right| < \epsilon.\]

But \( y(\epsilon) := (p_0 - 1)x_0 + p_1 x_1 + \cdots + p_k x_k \in C(A) \) and thus \( \lim_{\epsilon \to 0} y(\epsilon) = -x_0 \in C(A) \). Q.E.D.

**Example.** Let \( X_1, X_2, \cdots \) be a sequence of independent, identically distributed \( n \)-dimensional random vectors and \( S_r := X_1 + \cdots + X_r \) \((r = 1, 2, \cdots)\) the associated random walk. An \( x \in E^n \) is called possible if \( P(|S_r - x| < \epsilon \) for some \( r \)) > 0 \) for each \( \epsilon > 0 \) \(([2], [1])\). The set \( C \) of all possible \( x \) is obviously the closure of the additive semigroup generated by the set

\[A := \{ x \in E^n : P(|X_1 - x| < \epsilon) > 0 \text{ for each } \epsilon > 0 \}.\]

\( A \) is omnilateral, if \( X_1 \) has expectation 0. Indeed, if \( A \) is not omnilateral there exists a \( u \in E^n \) with \( ux \geq 0 \) for all \( x \in A \) and \( uy > 0 \) for some \( y \in A \). Assuming that the expectation \( \epsilon \) of \( X_1 \) exists and denoting the distribution
function of $X_1$ by $F_{X_1}$, we get

$$ue = u \int_{E^n} (x_1, \ldots, x_n) dF_{X_1}(x_1, \ldots, x_n) = \int_{A} ux dF_{X_1}(x_1, \ldots, x_n) > 0,$$

i.e. $e \neq 0$.

Thus, by Theorem 2, the set of all possible $x$ of a random walk with expectation 0 is a group.

REFERENCES


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