GROUP EXTENSIONS AND DISCRETE SUBGROUPS OF LIE GROUPS
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Abstract. Let $\Gamma$ be a discrete uniform subgroup of a connected simply connected solvable Lie group $S$. It is shown how $S$ is essentially determined by $\Gamma$, using the point of view of group extensions.

Let $\Gamma$ be a discrete uniform subgroup of a connected simply connected solvable Lie group $S$. The main purpose of this work is to study how $S$ is essentially determined by $\Gamma$ from the point of view of group extensions.

In [5], Mal’cev proved that $\Gamma$ determines $S$ uniquely if $S$ is nilpotent. However, for a general solvable group, the situation seems less favorable. Nevertheless, L. Auslander (Theorem 2 in [1]) has obtained some results when $\Gamma$ is the fundamental group of a nilmanifold, and in the subsequent works of Auslander and Tolimieri ([2], [8], [9]), stronger results were proved by using the notation of semisimple splittings, from which one can obtain the generalization of the result in [1] to arbitrary discrete uniform subgroups $\Gamma$. It is the purpose of this paper to revisit and prove directly this generalization by making use of the theory of group extensions as Auslander originally did in [1].

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1. Let $G$ and $H$ be topological groups. An extension of $G$ by $H$ is a pair $(E, \pi)$ consisting of a topological group $E$ which contains $G$ as a closed normal subgroup and a continuous open homomorphism $\pi$ of $E$ onto $H$ whose kernel is $G$. Two extensions $(E_1, \pi_1)$ of $G$ by $H$ are said to be equivalent if there exists an isomorphism $\sigma:E_1 \rightarrow E_2$ of topological groups which leaves elements of $G$ fixed and is such that $\pi_2 \sigma = \pi_1$. If $(E, \pi)$ is an extension of $G$ by $H$, then this determines a homomorphism $\pi^0:H \rightarrow O(G)$, where $O(G)$ denotes the group $A(G)$ of all automorphisms of $G$ modulo the inner automorphism group $I(G)$ of $G$. If $(E_1, \pi_1)$ and $(E_2, \pi_2)$ are equivalent, then $\pi^0_2 = \pi^0_1$. For any homomorphism $\varphi:H \rightarrow O(G)$, let $\text{Ext}(G, H, \varphi)$ denote the set of all equivalence classes of the extensions $(E, \pi)$ of $G$ by $H$.

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with \( \pi^0 = \phi \). Let \([E, \pi]\) denote the equivalence class containing \((E, \pi)\). \( \phi \) is called the character of \((E, \pi)\).

If \( G \) is a connected abelian Lie group and if \( H \) is a connected simply connected Lie group, then every extension \((E, \pi)\) of \( G \) by \( H \) can be represented by a continuous factor set \([4]\). Hence if \( G \) is a vector group, then we may represent \( \text{Ext}(G, H, \phi) \) as the cohomology group \( H^2(H, G, \phi) \) of \( H \) with coefficients in the continuous \( H \)-module \( G \) as in \([6]\).

We remark also that if \( H \) is discrete, then the above representation of \( \text{Ext}(H, G, \phi) \) is still valid.

2. Let \( \Delta \) be a finitely generated torsion free discrete nilpotent group and let \( N(\Delta) \) be the unique connected simply connected nilpotent Lie group having \( \Delta \) as a uniform subgroup (see Mal'cev \([5]\)).

Then the embedding \( A(\Delta) \subseteq A(N(\Delta)) \) induces a homomorphism \( \xi: O(\Delta) \to O(N(\Delta)) \) and we have

**Lemma.** If \((D, \sigma)\) is an extension of \( \Delta \) by \( \Gamma \) with the character \( \phi \), then there are a unique (up to equivalence) extension \((D_*, \sigma_*)\) of \( \Gamma \) by \( \phi \) with the character \( \xi \circ \phi \) and also a homomorphism \( \alpha: D \to D_* \) such that the following diagram commutes:

\[
\begin{array}{ccc}
1 & \longrightarrow & \Delta \\
\downarrow & & \downarrow \alpha \\
1 & \longrightarrow & N(\Delta) \\
& & \downarrow \sigma \\
& & D_* \\
& & \longrightarrow \Gamma \\
& & \longrightarrow 1 \\
\end{array}
\]

Moreover, if \([D, \sigma] = [D', \sigma']\), then \([D_*, \sigma_*] = [D_*', \sigma_*']\).

From this lemma, we can define \( \Lambda: \text{Ext}(\Delta, \Gamma, \phi) \to \text{Ext}(N(\Delta), \Gamma, \xi \circ \phi) \) by \( \Lambda([D, \sigma]) = [D_*, \sigma_*] \).

3. Let \((S, \pi)\) be an extension of a connected simply connected nilpotent Lie group \( N \) by a connected simply connected solvable Lie group \( H \) with the character \( \psi: H \to O(N) \) and let \( L \) be a discrete subgroup of \( H \). Thus in the direct product \( S \times L \), define \( S^* = \{(s, l) \in S \times L | \pi(s) = i(l)\} \), where \( i: L \subseteq H \).

Now define \( \pi^*: S^* \to L, \alpha: S^* \to S \) by \( \pi^*(s, l) = l, \alpha(s, l) = S \), respectively. Then embedding \( N \) into \( S^* \) under \( n \to (n, 1) \), we see that \((S^*, \pi^*)\) is an extension of \( N \) by \( L \) with the character \( \psi \circ i \) and that the following diagram commutes:

\[
\begin{array}{ccc}
1 & \longrightarrow & N \\
\downarrow & & \downarrow \alpha \\
1 & \longrightarrow & S \\
& & \downarrow \pi \\
& & H \\
& & \longrightarrow 1 \\
\end{array}
\]

It is easy to see that \((S^*, \pi^*)\) is unique (up to equivalence) and that if \([S_1, \pi_1] = [S_2, \pi_2]\), then \([S_1^*, \pi_1^*] = [S_2^*, \pi_2^*] \).
Hence we may define $\Omega(N, i) : \text{Ext}(N, H, \psi) \rightarrow \text{Ext}(N, L, \psi \circ i)$ by $\Omega([S, \pi]) = [S^*, \pi^*]$. Then we have

**Lemma.** If $L$ is a discrete uniform subgroup of $H$, then $\Omega(N, i)$ is one-to-one.

**Proof.** Let $C$ be the center of $N$ and let $\psi_\circ : H \to A(C)$, $(\psi \circ i)_\circ : L \to A(C)$ be the representation induced from $\psi : H \to O(N)$, $\psi \circ i : L \to O(N)$, respectively.

Clearly $(\psi \circ i)_\circ = \psi_\circ \circ i$. Hence $\Omega(C, i) : \text{Ext}(C, H, \psi_\circ) \rightarrow \text{Ext}(C, L, \psi_\circ \circ i)$ is defined.

We next identify $\text{Ext}(C, H, \psi_\circ) \cong H^2(H, C, \psi_\circ)$ and $\text{Ext}(C, L, \psi_\circ \circ i) \cong H^2(L, C, \psi_\circ \circ i)$. Under this identification, $\Omega(C, i) = i^* : H^2(H, C, \psi_\circ) \rightarrow H^2(L, C, \psi_\circ \circ i)$ being induced by $i : L \subset H$. Noting that $C$ is a vector group, it follows from a result of van Est and Mostow (see [3] or [6]) that $i^*$ (and hence $\Omega(C, i)$) is one-to-one.

Now since $H$ is connected and since $L$ is finitely generated as a fundamental group of a compact solvmanifold, we see from [4] that the groups $\text{Ext}(C, H, \psi_\circ)$ and $\text{Ext}(C, L, \psi_\circ \circ i)$ operate on the sets $\text{Ext}(N, H, \psi)$ and $\text{Ext}(N, L, \psi \circ i)$, respectively, and that this operation is simply transitive. It is an easy calculation to see that $\Omega(N, i)$ is equivariant with respect to these operations. Hence that $\Omega(C, i)$ is one-to-one implies that $\Omega(N, i)$ is one-to-one, proving the theorem.

4. We now prove

**Theorem (L. Auslander).** Let $S_1$ and $S_2$ be connected simply connected solvable Lie groups with discrete uniform subgroups $D_1 \subset S_1$, $D_2 \subset S_2$, and assume that there is an isomorphism $\alpha : D_1 \rightarrow D_2$ with the following properties:

(i) If $N_i$ is the nilradical of $S_i$ ($i = 1, 2$), then $\alpha$ induces an isomorphism $\alpha_i : D_1 \cap N_i \rightarrow D_2 \cap N_2$.

(ii) If $\varphi_i : S_i/N_i \rightarrow O(N_i)$ ($i = 1, 2$) is the homomorphism induced from the group extension $1 \rightarrow N_i \rightarrow S_i \rightarrow S_i/N_i \rightarrow 1$, then the following diagram commutes:

$$
\begin{array}{ccc}
S_1/N_1 & \xrightarrow{\varphi_1} & O(N_1) \\
\downarrow & & \downarrow \\
S_2/N_2 & \xrightarrow{\varphi_2} & O(N_2)
\end{array}
$$

where two vertical maps are induced by $\varphi_i$.

Then $S_1$ is isomorphic with $S_2$.

**Proof.** We identify $D_1$ with $D_2$ through $\alpha$. Then under this identification, we can identify $N_1$ with $N_2$, $S_1/N_1$ with $S_2/N_2$, and $\varphi_1$ with $\varphi_2$. Thus
we let $N=N_i$, $H=S_i/N_i$, $\varphi=\varphi_i$ and let $\pi_i:S_i\to H$ be the projection. Then $[S_i, \pi_i] \in \text{Ext}(N, H, \varphi)$, $i=1, 2$ and we have the commutative diagram

$$
\begin{array}{c}
1 \\ \downarrow \\ 1 \\
\Delta \\ \downarrow \\ ND \\ \downarrow \\ ND/N \\ \downarrow \\
\pi_i \\ H \\ \downarrow \\ 1
\end{array}
$$

where $D=D_1=D_2$, $\Delta=N\cap D$ and $\Gamma=D/\Delta$.

Since by [7] $\Delta$ and $\Gamma=ND/N$ are uniform in $N$ and $H$, respectively, it follows that $\Lambda([D, \sigma])=\Omega([S_i, \pi_i])$, $i=1, 2$, where $\Lambda$ and $\Omega=\Omega(N, i)$ are defined as in §§2 and 3, respectively. But $\Omega$ is one-to-one by Theorem 1. Hence $[S_1, \pi_1]=[S_2, \pi_2]$, proving that two extensions $(S_1, \pi_1)$ and $(S_2, \pi_2)$ are equivalent.

**Bibliography**