ON SEMICONTINUOUS LINEAR LATTICES

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Abstract. Applying spectral theory, we proved that a linear lattice is continuous if and only if it is semicontinuous and uniformly complete. In this paper we give another proof without use of spectral theory.

Let $L$ be a linear lattice. A sequence of elements $x_v \in L \ (v=1, 2, \cdots)$ is called a uniform Cauchy sequence if there is $a \in L$ such that for any $\varepsilon > 0$ we can find $v_0$ for which $|x_{\mu} - x_v| \leq \varepsilon |a|$ for $\mu, v \geq v_0$. $L$ is said to be uniformly complete if every uniform Cauchy sequence is convergent. We can easily prove that every uniformly complete linear lattice is Archimedean, as done in [1].

A linear lattice $L$ is said to be semicontinuous if every element $x \in L$ is normalizable, i.e. if $\{x\}^\perp$ is a normal manifold for any $x \in L$. In [1] we proved the

Theorem. A linear lattice is continuous if and only if it is semicontinuous and uniformly complete.

We used spectral theory to prove it. In this paper we will give another proof without use of spectral theory.

Let $L$ be a linear lattice. If $L$ is continuous, then $L$ is semicontinuous by Theorem 6.15 of [2] and uniformly complete by Theorem 3.3 of [2]. Thus we will prove the converse.

We suppose that $L$ is semicontinuous and uniformly complete. First we prove that if a sequence $0 \leq x_v \uparrow^\infty_{v=1}$ is bounded, then there is $z \in L$ such that $[x_v]x \uparrow^\infty_{v=1}[z]x$ for $x \geq 0$. If $x_v \leq k$ for $v=1, 2, \cdots$, then setting

$$y_n = \sum_{v=1}^{n} \frac{1}{v} ([x_v] - [x_{v-1}])k \quad \text{for } n = 1, 2, \cdots,$$

where $x_0 = 0$, we obtain a uniform Cauchy sequence $y_n \ (n=1, 2, \cdots)$. Since $L$ is uniformly complete by assumption, there exists $z \in L$ such that $y_n \uparrow^\infty_{n=1} z$. Then $[y_n]x \uparrow^\infty_{n=1}[z]x$ for $x \geq 0$ by Theorem 5.26 of [2] because...
\[ \{z\} = \{y_n: n = 1, 2, \cdots\}. \] On the other hand, by Theorem 6.3 of [2] we have
\[ [y_n] = \sum_{v=1}^{n} ([x_v] - [x_{v-1}])[k] = [x_n] \quad \text{for } n = 1, 2, \cdots. \]

Now we suppose that \( 0 \leq x_v \leq x_{v-1} \). For \( \alpha \geq 0 \), since \((\alpha x_1 - x_1)^+ \leq x_1\), there exists \( z_\alpha \in L \) such that
\[ ([\alpha x_1 - x_1]^+)x \uparrow \infty z_\alpha x \quad \text{for } x \geq 0, \]
as proved above. It is clear that \( [z_0] = 0 \) and \( [z_\alpha] = [x_1] \) for \( \alpha > 1 \) because \( \alpha > 1 \) implies \( (\alpha - 1)x_1 \leq (\alpha x_1 - x_1)^+ \leq x_1 \). Since
\[ (\alpha x_1 - x_1)^+ \leq (\beta x_1 - x_1)^+ \quad \text{for } \alpha \leq \beta, \]
we have \( [z_\alpha] \leq [z_\beta] \) for \( 0 \leq \alpha \leq \beta \). Since \([\alpha x_1 - x_1]^+ \leq [z_\alpha] \), we have
\[ ([z_\beta] - [z_\alpha])((\alpha x_1 - x_1)^+) = 0 \quad \text{for } \alpha \leq \beta. \]
Thus \([(z_\rho) - [z_\alpha])((\alpha x_1 - x_1)^+ = 0, \]
\[ \quad \text{for } \alpha \leq \beta \text{ and } \rho = 1, 2, \cdots. \]

We consider a double sequence \( 0 = a_{\mu,0} < a_{\mu,1} < \cdots < a_{\mu, n_{\mu}} \) (\( \mu = 1, 2, \cdots \)) such that \( a_{\mu, v} - a_{\mu, v-1} < 1/\mu \) for \( \mu = 1, 2, \cdots, n_{\mu}, \) where \( a_{\mu, n_{\mu}} > 1 \), and \( a_{\mu, v} \) is a partial sequence of \( a_{\mu+1, v} \) \( (v = 1, 2, \cdots, n_{\mu+1}) \). Setting
\[ y_{\mu} = \sum_{v=1}^{n_{\mu}} a_{\mu, v}([z_{a_{\mu, v}}] - [z_{a_{\mu, v-1}}])x_1 \quad \text{for } \mu = 1, 2, \cdots, \]
we obtain a uniform Cauchy sequence \( y_{\mu} \) \( (\mu = 1, 2, \cdots) \) because
\[ |y_{\mu} - y_{\rho}| \leq (1/\mu)[z_{a_{\mu, v}}]x_1 = (1/\mu)(x_1)x_1 = (1/\mu)x_1 \quad \text{for } \rho \geq \mu. \]
Since \( L \) is uniformly complete by assumption, there exists \( y \in L \) such that \( \lim_{\mu \to \infty} y_{\mu} = y \). Since
\[ a_{\mu, v-1}([z_{a_{\mu, v}}] - [z_{a_{\mu, v-1}}])x_1 \leq ([z_{a_{\mu, v}}] - [z_{a_{\mu, v-1}}])x_1, \]
we have
\[ y_{\mu} \leq \sum_{v=1}^{n_{\mu}} ([z_{a_{\mu, v}}] - [z_{a_{\mu, v-1}}])x_1 = [z_{a_{\mu, n_{\mu}}}]x_1 = [x_1]x_1 = x_1 \]
because \([x_1] \leq [x_1]\). Thus \( y \leq x_\rho \) for \( \rho = 1, 2, \cdots. \)

We suppose \( 0 \leq z \leq x_1 \) for \( v = 1, 2, \cdots. \) Since \([\alpha x_1 - x_1]^+)(\alpha x_1 - x_1) = (\alpha x_1 - x_1)^+ \geq 0 \), we have
\[ \alpha ([z_{a_{\mu, v}}] - [z_{a_{\mu, v-1}}])x_1 \geq ([\alpha x_1 - x_1]^+)x_1 \geq ([\alpha x_1 - x_1]^+)z. \]
for \( v = 1, 2, \cdots \), and hence \( \alpha [z_\alpha] x_1 \geq [z_\alpha] z \). Thus

\[
\alpha ([z_\alpha] - [z_\beta]) x_1 \geq ([z_\alpha] - [z_\beta]) z \quad \text{for } \alpha \geq \beta,
\]

because \( ([z_\alpha] - [z_\beta])[z_\alpha] = [z_\alpha] - [z_\beta] \) by Theorem 5.24 of [2]. Since \( \alpha_{\mu,v-1} + 1/\mu \geq \alpha_{\mu,v} (v = 1, 2, \cdots, n_\mu) \), we have

\[
y_\mu + \frac{1}{\mu} x_1 \geq \sum_{v=1}^{n_\mu} \alpha_{\mu,v} ([z_{s_\mu,v}] - [z_{s_\mu,v-1}]) x_1 \\
\geq \sum_{v=1}^{n_\mu} ([z_{s_\mu,v}] - [z_{s_\mu,v-1}]) z = [x_1] z = z
\]

for \( \mu = 1, 2, \cdots \) because \( 0 \leq z \leq x_1 \) implies \( [z] \leq [x_1] \). Since \( L \) is Archimedean, \( \lim_{\mu \to \infty} (1/\mu) x_1 = 0 \), and we obtain \( y \geq z \). Therefore \( L \) is continuous by definition.

REFERENCES


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