ON TOEPLITZ OPERATORS WHICH ARE CONTRACTIONS

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Abstract. We prove that a Toeplitz contraction $T_\phi$ is completely nonunitary if $\phi$ is not a constant. As an application, it is noted that for such $T_\phi$, a functional calculus can be defined for all functions $u$ in $H^\infty$ of the unit disk.

For $1 \leq p \leq \infty$, we denote by $L^p$ the usual class of Lebesgue measurable functions on the unit circle $\gamma$ of the complex plane. We write $H^p$ for the closed subspace of $L^p$ of functions whose Fourier coefficients vanish on the negative integers. We denote by $P$ the orthogonal projection of $L^2$ onto $H^2$ and by $B(H^2)$ the space of bounded operators on $H^2$. For $\phi \in L^\infty$, we consider the Toeplitz operator $T_\phi \in B(H^2)$ defined by $T_\phi f = P(\phi f)$ for $f \in H^2$. Following Sz.-Nagy and Foias [2], we say a contraction $T_\phi$, $\| T_\phi \| \leq 1$, is completely nonunitary (c.n.u.) if $T_\phi$ has no nontrivial reducing subspaces restricted to which $T_\phi$ is unitary. We will use the fact that $T_{\phi^*} = T_\phi$ [1, p. 137].

Theorem. If $\phi \in L^\infty$, $\| \phi \|_\infty \leq 1$ and $\phi$ is not a constant (almost everywhere), then $T_\phi$ is c.n.u.

Proof. Suppose $T_\phi$ is not c.n.u. We will show that $\phi$ is constant.

Let $S$ be a nontrivial reducing subspace for $T_\phi$ on which $T_\phi$ is unitary. We may write $S = \{ f \in H^2 : \| T_\phi f \|_2 = \| f \|_2 = \| T_{\phi^*} f \|_2 \text{ for } n = 1, 2, \ldots \}$. Now, for $f \in S$, $\| f \|_2 = \| T_\phi f \|_2 = \| T_{\phi^*} f \|_2 \leq \| \phi f \|_2 \leq \| f \|_2$ and the resulting equality gives $\phi f \in H^2$. Similarly, $\overline{\phi^*} f \in H^2$ for $n \geq 0$ and $f \in S$. We may apply the F. and M. Riesz theorem [1, p. 82] to the equality $\| \phi f \|_2 = \| f \|_2$ for a nonzero $f \in S$ to conclude that $|\phi| = 1$ almost everywhere on $\gamma$. Thus, we write $S = \{ f \in H^2 : \phi^* f, \overline{\phi^*} f \in H^2 \text{ for } n \geq 0 \}$.

Let $M_z$ be the operator of multiplication by the coordinate function $z$. Then $\phi^* f \in H^2$ implies $z \phi^* f = \phi^* z f \in H^2$ and similarly for $\overline{\phi^*} z f$, i.e. $M_z S \subseteq S$. By Beurling's theorem [1, p. 79], there is a function $\psi \in H^\infty$, $|\psi| = 1$ almost everywhere, such that $S = \psi H^2$. Since $1 \in H^2$, $\psi$ is in $S$. Note that $\phi \psi = \psi f$ for some $f \in H^2$ (since $\phi \psi \in S = \psi H^2$). Hence for

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$n \geq 0$, we have

$$\int \phi z^n \, dz = \int (\phi \psi)(\bar{\psi}z^n) \, dz = \int (\psi f)(\bar{\psi}z^n) \, dz = \int fz^n \, dz = 0$$

which implies that $\phi \in H^\infty$. Similarly, $\bar{\phi} \in H^\infty$ implies $\phi$ is a constant.

**Corollary.** If $\|\phi\|_\infty \leq 1$ and if $\phi$ is not constant, then the map $u \mapsto u(T_\phi)$ from $H^\infty$ into $B(H^2)$ defined by

$$u(T_\phi) = \lim_{r \to 1^-} \sum_{k=0}^\infty C_k r^k T_\phi^k,$$

where $u(\lambda) = \sum_{k=0}^\infty C_k \lambda^k \in H^\infty$ is a norm decreasing homomorphism of the algebra $H^\infty$ into $B(H^2)$.

**Proof.** Apply the above theorem and Theorem 2.1, Chapter III of [2].

**References**


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