REDUCIBILITY OF ISOMETRIC IMMERSIONS

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Abstract. For i=1, 2, suppose that the connected riemannian manifold \( M_i \) possesses a codimension \( p_i \) euclidean isometric immersion whose first normal space has dimension \( p_i \) and whose type number is at least two at each point, and let \( N = \dim(M_1 \times M_2) + p_1 + p_2 \). In this note it is proven that if \( f \) is any isometric immersion from the riemannian product \( M_1 \times M_2 \) into euclidean \( N \)-space \( \mathbb{E}^N \), then there exists an orthogonal decomposition \( \mathbb{E}^N = \mathbb{E}^{N_1}_1 \times \mathbb{E}^{N_2}_2 \) together with isometric immersions \( f_1: M_1 \to \mathbb{E}^{N_1}_1 \) and \( f_2: M_2 \to \mathbb{E}^{N_2}_2 \) such that \( f = f_1 \times f_2 \).

An isometric immersion \( f \) from a riemannian product \( M_1 \times M_2 \) into \( N \)-dimensional euclidean space \( \mathbb{E}^N \) is said to be reducible if there is an orthogonal product decomposition \( \mathbb{E}^N = \mathbb{E}^{N_1}_1 \times \mathbb{E}^{N_2}_2 \) together with isometric immersions \( f_1: M_1 \to \mathbb{E}^{N_1}_1 \) and \( f_2: M_2 \to \mathbb{E}^{N_2}_2 \) such that \( f = f_1 \times f_2 \). It is known that if \( M_1 \) and \( M_2 \) are connected and their Riemann-Christoffel curvature tensors are nonzero almost everywhere, then every codimension two euclidean isometric immersion of \( M_1 \times M_2 \) is reducible [1], [5]. This note is devoted to a more general reducibility theorem.

To formulate the hypothesis for our theorem, we let \( V \) denote the tangent space to a riemannian manifold \( M \) at a point \( m \). The Riemann-Christoffel curvature tensor \( R \) at \( m \) can be regarded as an endomorphism of \( V \wedge V \) which is symmetric with respect to the inner product defined by the riemannian metric. We will say that \( M \) satisfies condition \( A(p) \) at \( m \) if there exist vectors \( u, v \in V \) such that \( R(u \wedge v) \) has rank at least \( 2p \). (Recall that \( R(u \wedge v) \) has rank \( 2p \) iff \( p \) is the largest integer such that \( R(u \wedge v)^\Lambda \cdot \ldots \cdot R(u \wedge v) \) (\( p \) times)\( \neq 0 \) [3, p. 55].)

Reducibility Theorem. For \( i=1, 2 \), let \( M_i \) be a connected riemannian manifold which satisfies condition \( A(p_i) \) almost everywhere, and let \( N = \dim(M_1 \times M_2) + p_1 + p_2 \). Then any isometric immersion of the riemannian product \( M_1 \times M_2 \) into \( \mathbb{E}^N \) is reducible.

Notice that if \( M_i \) possesses a codimension \( p_i \) euclidean isometric immersion \( f_i \) whose first normal space has dimension \( p_i \) and whose type number is at least two at each point (in the sense of Allendoerfer [2], [4, pp. 349–354]), then \( M_i \) satisfies condition \( A(p_i) \) everywhere. In this

Presented to the Society, August 10, 1971; received by the editors October 5, 1971.
Key words and phrases. Isometric immersion, riemannian product, type number.

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case the dimension $N$ in the Reducibility Theorem is clearly optimal, because if $g: E^N \to E^{N+1}$ is a cylindrical isometric immersion, $g \circ (f_1 \times f_2): M_1 \times M_2 \to E^{N+1}$ will seldom be reducible. Reducibility is an uncommon phenomenon except in the lowest possible codimension.

$M$ satisfies condition $A(1)$ wherever $R$ is nonzero, i.e. at the nonflat points of $M$. Thus an induction based on the Reducibility Theorem and the following lemma will yield a generalization of Theorem 1 in [5].

**Lemma 1.** If the riemannian manifold $M_i$ satisfies condition $A(p_i)$ at $m_i$ for $i=1, 2$, then $M_1 \times M_2$ satisfies condition $A(p_1 + p_2)$ at $(m_1, m_2)$.

To prove the lemma we note that the tangent space $V$ to $M_1 \times M_2$ at $(m_1, m_2)$ possesses an orthogonal direct sum decomposition $V = V_1 \oplus V_2$, where $V_i$ consists of the vectors tangent to $M_i$. We regard $V_1 \wedge V_2$ as a subspace of $V \wedge V$. If $R_i: V_i \wedge V_i \to V_i \wedge V_i$ is the curvature tensor of $M_i$ and $\pi_i: V \wedge V \to V_i \wedge V_i$ is the orthogonal projection, then

$$R = R_1 \circ \pi_1 + R_2 \circ \pi_2$$

is the curvature tensor of $M_1 \times M_2$. If condition $A(p_i)$ holds at $m_i$, then there exist vectors $u_i, v_i \in V_i$ such that $R_i(u_i \wedge v_i)$ has rank at least $2p_i$.

By (1),

$$R((u_1 + u_2) \wedge (v_1 + v_2)) = R_1(u_1 \wedge v_1) + R_2(u_2 \wedge v_2).$$

Since $R_i(u_i \wedge v_i) \in V_i \wedge V_i$, the sum on the right has rank at least $2(p_1 + p_2)$, which proves that condition $A(p_1 + p_2)$ is satisfied at $(m_1, m_2)$.

Let $n = \dim(M_1 \times M_2)$ and $p = p_1 + p_2$ so that $N = n + p$. An isometric immersion $f: M_1 \times M_2 \to E^N$ and a choice of orthonormal basis for the normal space to $M_1 \times M_2$ at $(m_1, m_2)$ determine $p$ second fundamental forms $\Phi^i$, $n + 1 \leq i \leq N$, at $(m_1, m_2)$. The $\Phi^i$'s are symmetric bilinear forms on the tangent space $V$ at $(m_1, m_2)$, and they determine symmetric endomorphisms $A^i$ of $V$ by

$$\langle A^i(u), v \rangle = \Phi^i(u, v) \quad \text{for } u, v \in V,$$

where $\langle \ , \ \rangle$ denotes the riemannian metric. The $A^i$'s in turn determine symmetric endomorphisms $A^i \wedge A^i$ of $V \wedge V$ which satisfy the Gauss equation

$$R = \sum_{i=1}^{N} A^i \wedge A^i.$$

**Lemma 2.** Suppose that the second fundamental forms $\Phi^i$ of an isometric immersion $f$ from a connected riemannian product $M_1 \times M_2$ into $E^N$ have the following property at every point $(m_1, m_2)$ in $M_1 \times M_2$:

$$(3) \quad \Phi^i(w_1, w_2) = 0 \quad \text{for all } w_1 \in V_1, w_2 \in V_2, n + 1 \leq i \leq N.$$

Then $f$ is reducible.
This lemma is proven in §2 of [5]. In order to use Lemma 2 to prove the Reducibility Theorem it suffices by continuity to show that hypothesis (3) holds at almost all points in $M_1 \times M_2$. By the hypothesis of the Reducibility Theorem and Lemma 1 it suffices to show that (3) holds at those points of $M_1 \times M_2$ at which condition $A(p)$ is satisfied.

Assume now that $(m_1, m_2)$ is a point in $M_1 \times M_2$ at which condition $A(p)$ holds. Then we can choose vectors $u$, $v$ in the tangent space $V$ at $(m_1, m_2)$ so that $R(u \wedge v)$ has rank at least $2p$. By equation (2),

$$R(u \wedge v) = \sum_{\lambda=n+1}^{N} A^\lambda(u) \wedge A^\lambda(v).$$

It follows that $R(u \wedge v)$ has rank exactly $2p$ and that the $2p$ vectors $A^\lambda(u)$, $A^\lambda(v)$, $n+1 \leq \lambda \leq N$, are linearly independent. Now let $u = u_1 + u_2$ and $v = v_1 + v_2$, where $u_i$, $v_i \in V_i$. Using (1) and (2) we see that

$$\sum_{\lambda=n+1}^{N} A^\lambda(u_1) \wedge A^\lambda(u_2) = R(u_1 \wedge u_2) = 0,$$

and we conclude that the $A^\lambda(u_1)$'s (and hence the $A^\lambda(u_2)$'s) lie in the subspace of $V$ generated by the $A^\lambda(u)$'s; in fact, by Cartan's lemma,

$$A^\lambda(u_1) = \sum_{\mu=n+1}^{N} c_{\lambda}^\mu A^\mu(u), \quad c_{\lambda}^\mu = c_{\lambda}^\mu,$$

where the $c_{\lambda}^\mu$'s are real numbers. Similarly we can show that the $A^\lambda(v_1)$'s and the $A^\lambda(v_2)$'s lie in the span of the $A^\lambda(v)$'s. After a possible change of orthonormal basis for the normal space we can arrange that

$$A^\lambda(u_1) = c_{\lambda} A^\lambda(u), \quad A^\lambda(u_2) = (1 - c_{\lambda}) A^\lambda(u),$$

where $c_1 = 1$ for $n+1 \leq \lambda \leq q$, $c_\lambda \neq 0$, 1 for $q+1 \leq \lambda \leq r$, and $c_\lambda = 0$ for $r+1 \leq \lambda \leq N$. Equations (1), (2), and (4) now imply that

$$\sum_{\lambda=n+1}^{\ell} c_{\lambda} A^\lambda(u) \wedge A^\lambda(v_2) = \sum_{\lambda=n+1}^{N} A^\lambda(u_1) \wedge A^\lambda(v_2) = R(u_1 \wedge v_2) = 0,$$

$$\sum_{\lambda=q+1}^{N} A^\lambda(v_1) \wedge (1 - c_{\lambda}) A^\lambda(u) = 0.$$

Therefore $A^{q+1}(v_2), \ldots, A^{\ell}(v_2)$ and $A^{q+1}(v_1), \ldots, A^{N}(v_1)$ are in the span of the $A^\lambda(u)$'s. But they are also in the span of the $A^\lambda(v)$'s, and since the $A^\lambda(u)$'s and the $A^\lambda(v)$'s are linearly independent we must have

$$A^\lambda(v_2) = 0 \text{ for } n+1 \leq \lambda \leq r, \quad A^\lambda(v_1) = 0 \text{ for } q+1 \leq \lambda \leq N.$$
In particular the vectors $A^q(v)$, $q+1 \leq \lambda \leq r$, must vanish, and since these vectors are linearly independent, $q=r$.

We adopt the following index conventions: $n+1 \leq a \leq q$, $q+1 \leq \rho \leq N$. Then $c_a=1$, $c_{\rho}=0$, and it follows from (4) and (5) that

$$
A^q(u_1) = A^a(u), \quad A^q(v_1) = A^a(v), \quad A^\rho(u_1) = A^\rho(v_1) = 0.
$$

Hence if $w_2 \in V_2$,

$$
\sum_a A^a(u) \wedge A^a(w_2) = \sum_{a=n+1}^{N} A^a(u_1) \wedge A^a(w_2) = R(u_1 \wedge w_2) = 0,
$$

$$
\sum_a A^a(v) \wedge A^a(w_2) = 0.
$$

Since the $2(q-n)$ vectors $A^a(u)$, $A^a(v)$ are linearly independent, we can conclude that $A^a(w_2)=0$. Similarly we can show that $A^\rho(w_2)=0$ when $w_1 \in V_1$. Therefore

$$
\Phi^a(w_1, w_2) = \langle w_1, A^a(w_2) \rangle = 0, \quad \Phi^\rho(w_1, w_2) = \langle A^\rho(w_1), w_2 \rangle = 0,
$$

for $w_1 \in V_1$, $w_2 \in V_2$. This establishes (3) at $(m_1, m_2)$ and finishes the proof of the Reducibility Theorem.

REFERENCES


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