ON THE NOETHERIAN-LIKE RINGS OF E. G. EVANS

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Abstract. It is shown that if a commutative ring with identity
R is nonnoetherian, then the polynomial ring in one indeterminate
over R has an ideal with infinitely many maximal prime divisors
(in the sense of Nagata).

Let R denote a commutative ring with 1, and for any ideal A of R, let
\( \mathcal{Z}(A) = \{ r \in R \mid \text{there exists } s \in R \setminus A \text{ such that } rs \in A \} \). (By “ideal” we
shall always mean ideal \( \neq R \). The notation \( R \setminus A \) denotes the set-comple-
ment of A in R.) \( \mathcal{Z}(A) \) is merely the set of zero-divisors on the R-module
\( R/A \) and is always a union of prime ideals of R. Evans [1] calls R a \( ZD \)-
ring (zero-divisor ring) if for any ideal A of R, \( \mathcal{Z}(A) \) is a union of finitely
many prime ideals. We shall prove here the following:

Theorem. R is noetherian if (and only if) the polynomial ring in one
indeterminate \( R[X] \) is a \( ZD \)-ring.

Evans has proved in [1] the 2 indeterminate analogue of this theorem
(which follows from the theorem) and the special case of the theorem for
R containing an infinite field.

A prime ideal P of R such that P is maximal with respect to the property
of being contained in \( \mathcal{Z}(A) \) is called a maximal \( N \)-prime (for Nagata-
prime) of A. Note that such a prime contains A and that \( \mathcal{Z}(A) \) is the union
of the maximal \( N \)-primes of A. (See [2] and [4] for a perspective on the
associated primes of an ideal.)

Proof of Theorem. Suppose R is not noetherian. Then there exists
a strictly ascending chain \( (0) < (a_1) < (a_1, a_2) < \cdots < (a_1, \cdots, a_n) < \cdots \)
of ideals of R. Let \( f_0 = X, f_1 = 1 + X, \cdots, f_i = 1 + f_0 f_1 \cdots f_{i-1}, \cdots \). We wish
to show that the ideal \( A = (a_1 f_1, a_2 f_1 f_2, \cdots, a_n f_1 \cdots f_n, \cdots) \) in \( R[X] \)
has an infinite number of maximal \( N \)-primes and hence has the property that
\( \mathcal{Z}(A) \) is not a finite union of prime ideals. We show first that each \( f_i \in \mathcal{Z}(A) \).
Since \( A \subseteq (f_1) \) and \( f_1 \) is a monic polynomial of positive degree in \( R[X] \),
it follows that \( A \cap R = (0) \). Hence \( a_1 \notin A \), so \( a_1 f_1 \in A \) implies that \( f_1 \in \mathcal{Z}(A) \).
Similarly, to show \( f_n \in \mathcal{Z}(A) \), we wish to show \( a_n f_1 \cdots f_{n-1} \notin A \). Consider

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the residue class ring \( R/(a_1, \ldots, a_{n-1}) = R' \). The image of the ideal \( A \) in \( R'[X] \) is generated by the elements \( a'_n f_1' \cdots f'_{n-1}, a'_{n+1} f_1' \cdots f'_{n+1}, \ldots \), where "'" denotes image in \( R'[X] \). It will suffice to show that

\[
(a'_n f_1' \cdots f'_{n-1}) \subseteq (a'_n f_1' \cdots f'_{n-1}, a'_{n+1} f_1' \cdots f'_{n+1}, \ldots);
\]

and since \( f_1' \cdots f'_{n-1} \) is a monic polynomial in \( R'[X] \), this is equivalent to showing that \( a'_n \notin (a'_n f_1', a'_{n+1} f_n f'_{n+1}, \ldots) \subseteq (f'_n) \). Since \( f'_n \) is a monic polynomial of positive degree in \( R'[X] \), we have \( (f'_n) \cap R' = (0) \). Thus \( a'_n \notin (a'_n f_1', a'_{n+1} f_n f'_{n+1}, \ldots) \); hence we have proved \( f_n \in \mathcal{Z}(A) \).

Consider now \( f_i \) and \( f_j \) for \( i \neq j \). Clearly no prime ideal of \( R[X] \) contains both \( f_i \) and \( f_j \). Since each \( f_i \) is in \( \mathcal{Z}(A) \) and hence is in some maximal \( N \)-prime of \( A \), it then follows that \( A \) has infinitely many maximal \( N \)-primes. Q.E.D.²

A Lasker ring is one for which every ideal is a finite intersection of primary ideals. Such rings have been studied by Krull [3], and Evans has observed that every Lasker ring is a ZD-ring. Thus a consequence of the above theorem is that \( R \) is noetherian if (and only if) \( R[X] \) is Lasker.

We can add one further bit of information on the relationship between the ZD and noetherian properties.

**Proposition.** If \( R \) is a ZD-ring and \( R_P \) is noetherian for every prime ideal \( P \) of \( R \), then \( R \) is noetherian.

**Proof.** By [2, Corollary 1.4] it suffices to show that every ideal \( A \) of \( R \) has only finitely many \( B_w \)-primes (a \( B_w \)-prime, or weak-Bourbaki prime, of \( A \) is a prime ideal \( P \) such that \( P \) is a minimal prime divisor of \( A \): \( x \) for some \( x \in R \)). If \( P \) is a \( B_w \)-prime of \( A \), then \( P \) is contained in a maximal \( N \)-prime of \( A \); and since \( R \) is a ZD-ring, \( A \) has only a finite number of maximal \( N \)-primes, say \( Q_1, \ldots, Q_n \). Moreover, \( P \) is a \( B_w \)-prime of \( A \) in \( R \) and \( P \subseteq Q_i \) imply \( P R_{Q_i} \) is a \( B_w \)-prime of \( AR_{Q_i} \) [2, Proposition 1.2]. Since \( R_{Q_i} \) is noetherian, \( AR_{Q_i} \) has only a finite number of \( B_w \)-primes. Hence \( A \) can have only finitely many \( B_w \)-primes.

**References**


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² We are indebted to the referee for suggesting a judicious choice of the \( f_i \)'s, thus considerably shortening our original proof.