NONUNIQUENESS OF COEFFICIENT RINGS
IN A POLYNOMIAL RING

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Abstract. An example is given of commutative rings B, C with 1 such that \( B \cong C \) but \( B[t] \ncong C[t] \), where \( t \) is an indeterminate.

Several authors \([1], [2], [3]\) have recently studied the question, if \( B[t] \cong C[t] \) (\( B, C \) are commutative rings with 1, \( t \) is an indeterminate), does \( B \cong C \) follow? A simple counterexample is given below.

Let \( R \) be the reals and let \( P, Q, t, U, V, W, X, Y, Z \) be indeterminates. Let \( A = R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1) = R[x, y, z] \). Let \( \phi: A^3 \to A \) by \( \phi(a, b, c) = ax + by + cz \). Then \( \phi \) splits: map \( a \) to \( a(x, y, z) \). \( E = \ker \phi \) is well known to be a rank 2 projective which is not free, and hence requires 3 generators (that \( E \) is not free may be deduced from the fact that the tangent bundle of the real 2-sphere has no nonvanishing continuous sections). The splitting of \( \phi \) shows that \( A^3 \cong E \oplus A \). If we pass to symmetric algebras, we obtain the isomorphisms

\[
S(A^3) \cong A[P, Q, t] \cong S(E) \oplus A \quad S(A) \cong S(E) \otimes_A A[t] \cong S(E)[t],
\]

and since \( E \cong A^3/(x, y, z)A \),

\[
S(E) \cong A[U, V, W]/(xU + yV + zW).
\]

Let \( B = A[P, Q] \) and \( C = A[U, V, W]/(xU + yV + zW) \). We have shown that \( B[t] \cong C[t] \). It remains only to show that \( B \ncong C \). Suppose \( h: B \cong C \). \( B \) and \( C \) are \( A \)-subalgebras of the polynomial ring \( B[t] = A[P, Q, t] \) over \( A \). It is easy to show that the only invertible elements of \( A \), hence of \( B[t] \), and therefore of \( B \) and \( C \), are the nonzero real numbers. Since \( R \) has no nontrivial automorphisms, \( h \) must be an \( R \)-isomorphism. It is easy to check that \( A \) is a formally real domain. If \( D \) is a formally real domain and \( T \) is an indeterminate over \( D \), the only solutions of \( X^2 + Y^2 + Z^2 = 1 \) in \( D[T] \) already lie in \( D \). Hence, the only solutions of this equation in \( B[t] \) lie in \( A \), and the same holds for \( B \) and \( C \). Thus, \( h(A) \subset A \), and \( h^{-1}(A) \subset A \). After composing \( h \) with the automorphism of \( B \) which agrees with \( h^{-1} \) on \( A \) and fixes \( P, Q \), we can assume that \( h \) is an \( A \)-isomorphism of \( B \) and \( C \).
graded $A$-algebra. It follows that there are two elements $c = c_0 + c_1 + \cdots$, $c' = c'_0 + c'_1 + \cdots$ (where $c_i$ or $c'_i$ is the $i$-form component of $c$ or $c'$) such that $C = A[c, c'] = A[c - c_0, c' - c'_0]$. It follows easily that $c_1$, $c'_1$ span the $A$-module of 1-forms of $C$. But this module is isomorphic to $E$, and $E$ requires three generators, a contradiction. Thus, $B \neq C$.

A similar example has been noted by M. P. Murthy (unpublished).

**BIBLIOGRAPHY**


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