A FACTORIZATION THEOREM FOR COMPACT OPERATORS

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Abstract. It is shown that every compact operator $T:E\rightarrow F$ between Banach spaces admits a compact factorization ($T=QP$ where $P:E\rightarrow c$ and $Q:c\rightarrow F$ are compact) through a closed subspace $c$ of the Banach space $c_0$ of zero-convergent sequences.

A (linear) operator $T:E\rightarrow F$ between Banach spaces is compact if $T$ transforms the unit ball of $E$ into a relatively compact subset of $F$. The author has recently shown [3, Corollary 2.10] that an operator $T: E\rightarrow F$ is compact if and only if there is a sequence $\lambda$ in $c_0$ and a sequence $\{a_n\}$ in the unit ball of the topological dual $E'$ of $E$ such that

$$\|Tx\| \leq \sup \{\lambda_n \| \langle x, a_n \rangle \|$$

for all $x$ in $E$.

Theorem. If $T:E\rightarrow F$ is a compact operator between Banach spaces, then there is a closed subspace $c$ of $c_0$ and compact operators $P:E\rightarrow c$ and $Q:c\rightarrow F$ with $T=QP$.

Proof. Suppose that $T:E\rightarrow F$ is compact. Then there is a sequence $\lambda$ in $c_0$ and a sequence $\{a_n\}$ in $E'$ such that for each $x$ in $E$

$$(\ast) \quad \|Tx\| \leq \sup \{\lambda_n^2 \| \langle x, a_n \rangle \|.$$ 

Let $P:E\rightarrow c_0$ be the compact operator defined by $P(x)=\{\lambda_n \langle x, a_n \rangle \}$. Let $c$ denote the closure of $P(E)$ in $c_0$. Let $D:c\rightarrow c_0$ be the compact operator defined by $D(\xi)=\{\lambda_n \xi_n \}$. Let $S:D(c)\rightarrow F$ be the (unique) bounded (by $(\ast)$, $\|S\|\leq 1$) operator such that $S(DPx)=T(x)$ for all $x$ in $E$. Let $Q=SD$. Then $T=QP$, where both $Q$ and $P$ are compact.

Remark 1. Grothendieck [1, Chapitre I, p. 165] has shown that a Banach space $E$ has the approximation property if, for every Banach space $F$ and every compact operator $T:F\rightarrow E$, there exists a sequence of finite rank operators $T_n:F\rightarrow E$ with $\|T_n-T\|\rightarrow 0$. This result together with our factorization theorem can be used to give an elementary proof

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of the following result of Grothendieck [1, Chapitre I, pp. 170–171]: If each closed subspace $c$ of $c_0$ has the approximation property, then every Banach space has the approximation property.

Remark 2. Lindenstrauss and Tzafriri [2, p. 265] have recently shown that a Banach space $E$ is isomorphic to a Hilbert space if and only if, for every closed subspace $F$ of $E$, every compact operator $T:F\to F$ can be extended to a bounded operator $S:E\to F$. By combining this result with our factorization theorem it follows that a Banach space $E$ is isomorphic to a Hilbert space if and only if, for every closed subspace $F$ of $E$ and every closed subspace $c$ of $c_0$, every compact operator $T:F\to c$ can be extended to a bounded operator $S:E\to c$. This result contrasts with the following (unpublished) result of the author: If $E$ is an infinite dimensional Hilbert space, then there exists a closed subspace $c$ of $c_0$ and a compact operator $T:c\to E$ that cannot be extended to a bounded operator from $c_0$ into $E$.

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References


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