A CHARACTERIZATION OF HEREDITARILY INDECOMPOSABLE CONTINUA

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Abstract. Hereditarily indecomposable continua are characterized by giving a condition which some defining sequence for the continua must satisfy.

In [1], Lida K. Barrett gave a necessary and sufficient condition, in terms of chains, for a chainable continuum to be indecomposable. W. T. Ingram and H. Cook [2] obtained a more general theorem by giving a characterization of all indecomposable continua by using coherent collections. An incorrect characterization, in terms of chains, of hereditarily indecomposable chainable continua was given in [1] and was corrected by A. Lelek in his review [MR 24, #A2948] of that paper. Now a necessary and sufficient condition is presented for continua, not just those which are chainable, to be hereditarily indecomposable by referring to coherent collections of open sets in the statement of the condition.

All sets will be assumed to be subsets of compact metric spaces. Each compact continuum $M$ is known to have a sequence $G_1, G_2, \ldots$ of finite open covers of $M$ such that, for each natural number $n, (1)$ the mesh of $G_n$ is less than $1/n$, (2) each open set in $G_n$ contains a point of $M$ that is not contained by any other set in $G_n$, and (3) each open set in $G_{n+1}$ has its closure contained as a subset of some element of $G_n$. A sequence of open covers with the above properties is called a defining sequence for $M$. Whenever $T$ represents a collection of sets, the symbol $T^*$ will represent the union of the elements of $T$. A collection $T$ of sets is said to be coherent if and only if it is true that, whenever $T$ is expressed as the union of two subcollections $T_1$ and $T_2$, then $T^*_1$ and $T^*_2$ have a point in common. A chain is a finite collection $\{c_1, c_2, \ldots, c_n\}$ of open sets such that, for $1 \leq i < j \leq n$, $c_i$ intersects $c_j$ if and only if $i = j - 1$. The elements of a chain are referred to as links. The definitions of all other terms used in this paper can be found in [3].

Theorem. A compact continuum $M$ is hereditarily indecomposable if and only if there is a defining sequence $G_1, G_2, \ldots$ for $M$ such that, for each
positive integer \( i \), it is true that if \( A \) and \( B \) are elements of \( G \) with disjoint closures then there is an integer \( j > i \) having the property that if \( L \) is a coherent collection of open sets of \( G \) with \( L^* \) intersecting both \( A \) and \( B \) then whenever \( L \) is expressed as the union of two coherent collections \( L_1 \) and \( L_2 \) there is an element \( C \) in \( G \) such that the elements of \( L_1 \) or \( L_2 \) together with \( C \) form a coherent collection the union of whose elements intersects both \( A \) and \( B \).

**Proof.** Suppose \( M \) is a compact continuum which has a defining sequence as described in the statement of the theorem. Assume that \( M \) has a subcontinuum \( M' \) which is the union of two proper subcontinua \( H \) and \( K \). Choose a point \( x \) from \( H—K \) and a point \( y \) from \( K—H \). There is a positive integer \( i_0 \) such that each \( P \) and \( Q \) in \( G_{i_0} \) which contains \( x \) and \( y \), respectively, have disjoint closures and each chain \( E \) of open sets of \( G_{i_0} \) with no more than three links has the property that if \( E^* \) contains \( x \) then \( E^* \) does not intersect \( K \) and if \( E^* \) contains \( y \) then \( E^* \) does not intersect \( H \). Let \( A \) and \( B \) denote two elements of \( G_{i_0} \) which contain \( x \) and \( y \), respectively. Choose any natural number \( j_0 > i_0 \). The collection \( L = \{ g \in G_{j_0} : g \) intersects \( M' \} \) is a coherent subcollection of \( G_{i_0} \) such that \( L^* \) intersects \( A \) and \( B \). The union of the two coherent collections \( L_1 = \{ g \in G_{j_0} : g \) intersects \( H \} \) and \( L_2 = \{ g \in G_{j_0} : g \) intersects \( K \} \) is the collection \( L \), and \( G_{i_0} \) does not contain an element \( C \) such that \( L_1 \cup \{ C \} \) or \( L_2 \cup \{ C \} \) is a coherent collection with elements intersecting \( A \) and \( B \). This contradiction shows that \( M' \) must be indecomposable; therefore, \( M \) is hereditarily indecomposable.

Suppose \( M \) is a hereditarily indecomposable continuum and \( G_1, G_2, \ldots \) is any defining sequence for \( M \). Assume that, for some integer \( i_0 \), there are elements \( A \) and \( B \) of \( G_{i_0} \) which have disjoint closures and which have the property that, for each natural number \( k \), a coherent subcollection of \( G_{i_0+k} \) is the union of two coherent collections \( L_{k1} \) and \( L_{k2} \) such that both \( A \) and \( B \) are intersected by \( L_{k1}^* \cup L_{k2}^* \) and if \( C \) is an element of \( G_{i_0} \) then neither \( L_{k1} \cup \{ C \} \) nor \( L_{k2} \cup \{ C \} \) is a coherent collection which has the union of its elements intersecting both \( A \) and \( B \). Notation is now chosen such that \( L_{k1}^* \) intersects \( A \) and \( L_{k2}^* \) intersects \( B \) for all natural numbers \( k \). Due to \( L_{k1}^* \) intersecting \( A \) for each \( k \), there is a point \( x_0 \) of \( A \) which is also a point of \( \bigcap_{n=1}^{\infty} \cl(\bigcup_{k=n}^{\infty} L_{k1}^*) \). A sequence of points \( x_1, x_2, \ldots \) of \( M \) can be found such that the sequence converges to \( x_0 \) and such that there is a subsequence \( k_1, k_2, \ldots \) of natural numbers with \( x_i \in L_{k_i}^* \) for each natural number \( i \). The set \( L_{k_i}^* \) intersects \( B \) for each \( i \); thus, there is a point \( y_0 \) of \( B \) and a subsequence \( m_1, m_2, \ldots \) of \( k_1, k_2, \ldots \) such that a sequence of points \( y_1, y_2, \ldots \) of \( M \) converges to \( y_0 \) with \( y_i \in L_{m_i}^* \) for each \( i \). Define \( M_1 \) to be the set \( \bigcap_{i=1}^{n-1} \cl(\bigcup_{j=i}^{n-1} L_{m_i}^*) \) and \( M_2 \) to be the set \( \bigcap_{i=1}^{n-1} \cl(\bigcup_{j=i}^{n-1} L_{m_i}^*) \). Notice that \( M_1 \) contains \( x_0 \) and \( M_2 \) contains \( y_0 \). Since each point which is a point of \( M_1 \) or \( M_2 \) is of distance zero from \( M \), \( M_1 \) and \( M_2 \) are subsets of \( M \). The
point \( x_0 \) does not belong to \( M_2 \); otherwise, an element \( D \) of \( G_{i_x} \) that contains \( x_0 \) would intersect \( L_{m_i}^* \) for some \( i \), which would imply that \( L_{m_i}^* \cup \{D\} \) would be a coherent collection with one of its elements intersecting \( A \), namely \( D \), and one of its elements intersecting \( B \). Similarly, \( M_1 \) does not contain \( y_0 \). Since \( L_{m_1}^* \) and \( L_{m_2}^* \) intersect for each \( i \), \( M_1 \) and \( M_2 \) have a common point. We now show that \( M_1 \) and \( M_2 \) are each connected. Suppose \( M_x = H \cup K \) where \( H \) and \( K \) are mutually exclusive closed sets. Suppose \( x_0 \) belongs to \( H \). There are open sets \( D_H \) and \( D_K \) containing \( H \) and \( K \) as subsets, respectively, such that \( D_H \) does not intersect \( D_K \). Since \( D_H \) contains \( x_0 \), \( D_H \) contains all but at most finitely many of the terms of the sequence \( x_1, x_2, \ldots \); thus, \( D_H \) intersects all but at most finitely many of the sets \( L_{m_1}^*, L_{m_2}^*, \ldots \). Since \( D_K \) contains a point of \( M_1 \), \( D_K \) intersects infinitely many of the sets \( L_{m_1}^*, L_{m_2}^*, \ldots \). This shows that there is a subsequence \( n_1, n_2, \ldots \) of \( m_1, m_2, \ldots \) such that each of \( D_H \) and \( D_K \) intersects \( L_{m_i}^* \) for each \( i \). There is an \( m_0 \) such that each chain of elements of \( G_m \) (\( m \geq m_0 \)) has more than three links if one of its links intersects \( D_H \) and one of its links intersects \( D_K \). This shows that there is a sequence of open sets \( g_1, g_2, \ldots \) such that \( g_i \in L_{n_{m_i}}^* \) and \( g_i \) does not intersect \( D_H \) or \( D_K \) for each \( i \). Letting \( z_i \in g_i \) for each \( i \), there is a limit point of the set \( \{z_1, z_2, \ldots \} \) which is a point of \( M_1 \) and which is not a point of \( D_H \) or \( D_K \). This contradiction proves that \( M_1 \) is connected. The set \( M_2 \) is connected by a similar argument. Now \( M_1 \cup M_2 \) has been proven to be a decomposable subcontinuum of \( M \). We must conclude that \( G_1, G_2, \ldots \) has the property as described in the theorem.

REFERENCES


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