RECOGNIZING MANIFOLDS AMONG GENERALIZED MANIFOLDS

DENNIS C. HASS

Abstract. This paper provides various conditions, on the complement of a point in a generalized manifold $M$, which imply that $M$ is a classical topological manifold. Similar characterizations are given for $m$-spheres and 3-cells.

This paper announces a few results in the classic quest for a property which characterizes the topological manifolds among the generalized manifolds.

It is well known, for example, that if a 3-gm $M$ is a product space then it is a manifold. In [1], Raymond showed that the factors are generalized manifolds. Further, Wilder [2] says these factors are manifolds; thus, $M$ is too. Clearly, then a 3-gm which is locally a product space is also a manifold.

Efforts have been made to weaken this hypothesis. In [3] K.W. Kwun and F. Raymond proved that a 3-gm which is locally conical is a manifold. $M$ is locally conical if for all $P$ in $M$, $P$ has a neighborhood $N$ such that $N—P=E^1×bN$. Our results show that one need not specify the factors in advance. By assuming, only, that $N—P$ is any product space, we can show that $M$ is still a manifold; see Theorem 3.

Lemma 1. If $M$ is a connected $m$-gm, for $m≥2$ such that for $P$ in $M$, $M—P=A×B$ is a product space, then each of $M—P$, $A$, and $B$ is homologically trivial up through dimension $m—2$.

Proof. Proof of these claims is exactly analogous to Chapter 3 of the author’s dissertation [4], except for one minor change. We replace the fact that $π_kA×B=π_kA×π_kB$ with the Künneth formula and an induction on $k$. It is also still true, that if $B$ is compact, then $M—P=E^1×B$.

Let $M$ be a connected 3-gm.

Theorem 2. For $P$ in $M$, if $M—P$ is a product space, then $M$ is either $S^3$ or $E^3$.

Received by the editors August 25, 1971.


Key words and phrases. Topological manifolds, classical manifolds, generalized manifolds, locally conical generalized manifold, spheres, cells.

© American Mathematical Society 1972

311
Proof. By Lemma 1, the factorization of $M - P$ is either $E^1 \times E^2$ or $E^1 \times S^3$, respectively.

Theorem 3. If for each $P$ in $M$, there is an open neighborhood $N$ of $P$ such that $N - P$ is a product space, then $M$ is a classical 3-manifold.

Proof. If $N$ is compact, then $N = M$ and $M = S^3$, by Theorem 2. If $N$ is not compact, then $N = E^3$ by Theorem 2.

Now, let $M$ be a connected $m$-gm, for $m \geq 4$.

Theorem 4. If for $P$ in $M$, $M - P$ is a product space, then either $M - P$ is homologically trivial or $M - P$ is $E^1$ times a generalized $(m-1)$-sphere.

Proof. By Lemma 1, each of $M - P = A \times B$, and $A$, and $B$ is $(m-2)$-connected with respect to homology. If neither $A$ nor $B$ is compact, then each is homologically trivial and of course so is $M - P$. If $B$ is compact, then as mentioned in the proof of Lemma 1, we have $M - P = E^1 \times B$. Since $B$ is closed and $(m-2)$-connected (homology) the theorem follows.

Theorem 5. If for $P$ in $M$, $M - P$ is a product of (many?) factors each of dimension 2 or less, then $M = S^m$.

Proof. In view of Lemma 1, we may assume that none of the factors is compact. According to Theorem 4, each factor is either $E^1$ or $E^2$.

Next, let $M$ be a compact connected $m$-gm, for $m \geq 5$.

Theorem 6. For $P$ in $M$, if $M - P$ is a product of simply-connected (homotopy fundamental group is trivial) PL manifolds $A$ and $B$, then $M = S^m$.

Proof. Using Theorem 4, $M - P$, $A$, and $B$ are each $(m-2)$-connected (homotopy this time!). If neither $A$ nor $B$ is compact, then each is contractible.

J. Stallings [5] proved that in this case $M - P = E^m$. Finally, let $M$ be a connected $m$-gm, for $m \geq 6$.

Theorem 7. If for all $P$ in $M$, there is a neighborhood $N$ of $P$ such that $N - P = A \times B$ is a product of simply-connected manifolds $A$ and $B$, then $M$ is a classical $m$-manifold.

Proof. By Theorem 4, each of $N - P$, $A$, and $B$ is $(m-2)$-connected (homotopy!). In light of Theorem 6, we may assume that $B$ is compact. By Lemma 1, $N - P = E^1 \times B$ and $B$ is a homotopy $(m-1)$-sphere. Since $m - 1 \geq 5$ we have $B = S^{m-1}$ by the Poincaré theorem. Thus, $N = E^m$ as desired.

That $N - P = A \times B$ inherits the manifold property from $A$ and $B$ is not new; it is new that the homology groups of $A$ and $B$ may be calculated.
and need not be assumed. Note that none of the results here or in [4] relies on the unproven Poincaré conjectures.

ACKNOWLEDGEMENT. I want to thank the Mathematics Department of Randolph-Macon Woman’s College for financial support while I studied this topic.

BIBLIOGRAPHY


Department of Mathematics, Randolph-Macon Woman’s College, Lynchburg, Virginia 24504