ON FINITE INVARIANT MEASURES FOR SETS OF MARKOV OPERATORS

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Abstract. A. Brunei [1] proved that a Markovian operator $P$ has an invariant measure if and only if each convex combination of iterates $\sum_{n=0}^{\infty} \alpha_n P^n$ is conservative. In the present paper this result is generalized for any commutative semigroup of Markovian operators:

Let $\Pi$ be a semigroup; there exists a common invariant measure for $\Pi$ if and only if each convex combination $\sum_{n=1}^{\infty} \alpha_n P_n$, where $(\{P_n\}) \subset \Pi$, is conservative.

1. Definitions and notations. Let $(X, \Sigma, m)$ be a finite measure space. A Markov operator $P$ is a positive contraction on $L_1(X, \Sigma, m)$ i.e., (i) $\|P\| \leq 1$, (ii) $u \geq 0 \Rightarrow uP \geq 0$. We shall use the notations of [3], so the operator adjoint to $P$ which is defined in $L_\infty(m)$ will also be denoted by $P$ to the left side of the variable. Thus $\langle uP, f \rangle = \langle u, Pf \rangle$, $u \in L_1(m), f \in L_\infty(m)$. We denote $\Sigma_i(P) = \{ A \in \Sigma | P1_A = 1_A \text{ a.e.} \}$. If $P$ is conservative, then $\Sigma_i(P)$ is a field.

Let us consider the commutative semigroup $\Pi$ of conservative Markov operators. The invariant sets of $\Pi$ is the collection $\Sigma_i = \bigcap_{P \in \Pi} \Sigma_i(P)$.

Let $\Pi$ be the convex hull of $\Pi$: (i) $\Pi \subset \Pi$, (ii) $\{P_n\} \subset \Pi$ and $P_n \to P$ in operator norm then $P \in \Pi$. (iii) $P_1, P_2 \in \Pi$, $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$ then $\alpha P_1 + \beta P_2 \in \Pi$. (iv) $\Pi$ is minimal under those conditions.

A measure $\mu$ is said to be invariant for $\Pi$ if $\mu P = \mu$ for every $P \in \Pi$. Clearly, $\Pi$ is a semigroup and if $\mu$ is a finite invariant measure for $\Pi$ it is invariant for $\Pi$.

In this paper we prove that if there is no finite invariant measure for $\Pi$ then there exists an operator $Q \in \Pi$ such that the dissipative part of $Q$, $D = X$. (For definitions see [3].)

Remark. In [1] it is proved for $\Pi = \{P^n\}$, where $P$ is a given Markov operator, that if it has no finite invariant measures, then there exists $Q \in \Pi$, such that the dissipative part of $Q$, $D \neq \emptyset$.

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2. Conservative operators and invariant measures.

**Lemma 2.1.** If there are no finite invariant measures for \( \Pi \), then there exists \( \{P_n\} \subset \Pi \) and \( 0 \leq f \in L_\infty(m) \) such that \( \sum_{n=1}^{\infty} P_n f \in L_\infty(m) \).

**Proof.** In [4] it is proved that if there exists no finite invariant measure then there exists \( 0 \leq g \in L_\infty(m) \) such that \( \inf \{ \int P_g \, dm \mid P \in \Pi \} = 0 \). Hence by slight modifications of the proof of Lemma C, Chapter IV of [3], or of the more elegant proof of this lemma which appears in [2], Lemma 2.1 can be proved.

Let us define the space:

\[
L = \text{span}\{ (I - P)L_\infty(m) \mid P \in \Pi \}.
\]

The orthogonal complement of \( L \) is

\[
L^\perp = \{ v \in L_\infty^*(m) \mid vP = v \ \forall P \in \Pi \}.
\]

\( L_\infty^*(m) \) is the space of the charges (finitely additive measures). If \( P1 = 1 \), as it is in the conservative case, then \( vP = v \) implies \( v^+ P = v^+ \). Define:

\[
M = \{ v \in L_\infty^*(m) \mid v \geq 0, \| v \| = 1, vP = v \ \forall P \in \Pi \}.
\]

It is easy to show that:

\[
f \in L \iff \langle v, f \rangle = 0, \quad v \in M.
\]

**Lemma 2.2.** If there exists no finite invariant measures for \( \Pi \), then there exists \( 0 \neq f \geq 0 \) such that \( f \in L \).

**Proof.** Let \( f \) be the function of Lemma 2.1. It is clear that \( \langle v, f \rangle = 0 \) or each \( v \in M \) and by (2.4), \( f \in L \).

**Lemma 2.3.** \( X \) may be decomposed uniquely into the disjoint union \( X = X_0 \cup X_1 \) where (i) \( X_0, X_1 \in \Sigma_i \). (ii) There exists a finite invariant measure for \( \Pi \) equivalent to \( m_{|X_1} \). (iii) There exists \( \{A_n\} \subset \Sigma \) with \( A_n \not\subset X_0 \), and \( 1_{A_n} \in L, \forall n \).

**Proof.** Let \( \mu \) be any finite invariant measure for \( \Pi \), let \( B = \text{supp } \mu \); it is easy to see that \( B \in \Sigma_i \). Let \( \alpha = \sup \{ m(B) \mid B = \text{supp } \mu, \mu \text{ a finite invariant measure for } \Pi \} \). Hence there exists a sequence of finite invariant measures \( \{\mu_n\} \), such that \( m(B_n) \not\subset \alpha_n \), where \( B_n = \text{supp } \mu_n \). Define \( X_1 = \bigcup_{n=1}^{\infty} B_n \) and \( \lambda = \sum_{n=1}^{\infty} (1/2^n) \mu_n \) and then \( \lambda \) is a finite invariant measure with \( \text{supp } \lambda = x_1, \ m(X_1) = \alpha \), and \( X_1 \in \Sigma_i \) (or \( P1_{X_1} = 1_{X_1}, \forall P \in \Pi \)). Define \( X_0 = X - X_1 \), assume that there exists a finite invariant measure for \( \Pi \), \( \lambda' \) supported on \( X_0 \). Let \( \text{supp } \lambda' = B' \subset X_0 \), then \( \lambda + \lambda' \) is a finite invariant measure for \( \Pi \), \( \text{supp}(\lambda + \lambda') = X_1 \cup B' \) and \( m(X_1 \cup B') > \alpha \), a contradiction.
Since $X_0 \in \Sigma_i$ (and $P_1X_0 = I_{X_0}$) we can restrict the Markov operators of $\Pi$ to $(X_0, \Sigma_{X_0}, m_{X_0})$, and apply Lemma 2.2. Formula (2.4) implies that if $0 \leq f \leq g \leq f \in L$ and $f \in L$, then $\max(f, g) \in L$. Let $\mathcal{A} = \{A | A \in \mathcal{L}\}$; clearly if $0 \leq f \leq g \in L$ and $A = \{f \geq \varepsilon > 0\}$ then $A \in \mathcal{A}$, if $A \in \mathcal{A}$ and $B \subset A$ then $B \in \mathcal{A}$, and if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. Let $\beta = \sup \{m(A) | A \in \mathcal{A}\}$. There exists a sequence $\{A_n\}$ such that $m(A_n) \geq \beta$. It can be supposed that $A_n \subset A_{n+1}, \forall n$ (if not, replace $A_n$ by $A'_n = \bigcup_{k=1}^n A_n$, and then $A'_n \in \mathcal{A}$ and $m(A'_n) \geq \beta$). Let $A_n \not\subset X'_0$. If $X'_0 \not\subset \sum_i$ then there exists $\nu \in \Pi$ such that $\{P_1X'_0 \geq \varepsilon \} \not\subset X'_0$, and hence $X_n, \varepsilon > 0$ such that $\{P_1A_n > \varepsilon \} \not\subset X'_0$. Let $E = \{P_1A_n > \varepsilon \}$, $1_E \in L$ by (2.4), because for each $\nu \in M$ we have:

$$\langle \nu, 1_E \rangle \leq (1/\varepsilon) \cdot \langle \nu, P_1A_n \rangle = (1/\varepsilon) \cdot \langle \nu, 1_A_n \rangle = 0.$$ 

Denote $A'_n = A_n \cup E$, $A'_n \in \mathcal{A}$ and $m(A'_n) \not\geq \beta$, a contradiction, hence $X'_0 \subset \sum_i$.

If $X'_0 \not\subset X_0$, then we can restrict the Markov operators of $\Pi$ to $(X_0 - X'_0, X_0, \Sigma_{X_0 - X'_0}, m_{X_0 - X'_0})$, and by Lemma 2.2 there exists $\mathcal{A} \supset E \subset X_0 - X'_0$, denote $A'_n = A_n \cup E$, $A'_n \in \mathcal{A}$ and $m(A'_n) \not\geq m(X_0 - E) > \beta$, a contradiction. So, $X'_0 = X_0$ and Lemma 2.3 is proved.

**Lemma 2.4.** Let $\{A_n\}$ be the sequence of Lemma 2.3, part (iii), then for each $n$ and for each $\varepsilon > 0$, there exists an operator $Q \in \Pi$ such that $\|Q_1A_n\| < \varepsilon$.

**Proof.** $A_n \in L$, hence there exist $f_1, f_2, \ldots, f_j \in L_\infty$ and $P_1, P_2, \cdots, P_j \in \Pi$ such that

$$\|f_1 - P_1f_1\| + \|f_2 - P_2f_2\| + \cdots + \|f_j - P_jf_j\| - A_n < \varepsilon/2.$$

Hence:

$$\left\| \frac{1}{N_j} \sum_{i=1}^N \sum_{i=1}^N \cdots \sum_{i=1}^N P_1^{i_1}P_2^{i_2} \cdots P_j^{i_j} \right\|_\infty$$

$$\leq \left\| \frac{1}{N_j} \sum_{i=1}^N \sum_{i=1}^N \cdots \sum_{i=1}^N P_1^{i_1}P_2^{i_2} \cdots P_j^{i_j} (f_1 - P_1f_1) \right\|_\infty$$

$$+ \left\| \frac{1}{N_j} \sum_{i=1}^N \sum_{i=1}^N \cdots \sum_{i=1}^N P_1^{i_1}P_2^{i_2} \cdots P_j^{i_j} (f_2 - P_2f_2) \right\|_\infty$$

$$+ \left\| \frac{1}{N_j} \sum_{i=1}^N \sum_{i=1}^N \cdots \sum_{i=1}^N P_1^{i_1}P_2^{i_2} \cdots P_j^{i_j} (f_j - P_jf_j) \right\|_\infty$$

$$\times [(f_1 - P_1f_1) + (f_2 - P_2f_2) + \cdots + (f_j - P_jf_j) - A_n] \right\|_\infty.$$
But

\[
\left\| \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_j=1}^{N} \prod_{i=1}^{j} p_{i_1}^i p_{i_2}^i \cdots p_{i_k}^i (f_k - P_k f_k) \right\|_\infty
\]

\[
= \left\| \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{j-1}=1}^{N} \sum_{i_{j+1}=1}^{N} \prod_{i=1}^{j} p_{i_1}^i p_{i_2}^i \cdots p_{i_{j-1}}^i p_{i_{j+1}}^i \cdots p_{i_k}^i \right\|_\infty
\times \left[ \frac{1}{N} \sum_{i_{j+1}=1}^{N} p_{i_{j+1}}^i (f_k - P_k f_k) \right]_\infty
\]

\leq 2 \|f_k\|_\infty/N,

if \(N\) is sufficiently large then \(2 \|f_k\|_\infty/N \leq \epsilon/2j\) for \(1 \leq k \leq j\). Let

\[
Q = \frac{1}{N^j} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_j=1}^{N} \prod_{i=1}^{j} p_{i_1}^i p_{i_2}^i \cdots p_{i_k}^i
\]

and then \(\|Q1_{A_n}\| < \epsilon\).

**Lemma 2.5.** Let \(\{A_n\}\) be the sequence of Lemma 2.3, part (iii), then there exists an operator \(V \in \Pi\) such that \(\lim_{k \to \infty} \|V^k1_{A_n}\| = 0, \forall n\).

**Proof.** By Lemma 2.4 it can be shown that there exists a sequence of operators \(\{Q_n\} \subset \Pi\) such that \(\|Q_n1_{A_n}\| \leq 1/n\). Let \(\alpha_n\) be positive numbers such that \(\sum_{n=1}^{\infty} \alpha_n = 1\). Let \(V = \sum_{n=1}^{\infty} \alpha_n Q_n\), then \(V \in \Pi\). Given an integer \(N\), denote \(\beta = \sum_{n=1}^{N} \alpha_n\), \(\gamma = \sum_{n=N+1}^{\infty} \alpha_n\), \(\beta + \gamma = 1\). Define the operators 

\[
R = (1/\beta) \sum_{n=1}^{N} \alpha_n Q_n\text{ and } S = (1/\gamma) \sum_{n=N+1}^{\infty} \alpha_n Q_n, \text{ } R, S \in \Pi, \beta R + \gamma S = V, \text{ and } \|S1_{A_N}\| \leq 1/N.\n\]

Hence

\[
\|V^k1_{A_N}\| = \|(\beta R + \gamma S)\|^k1_{A_N}\| \leq \beta^k \|R^k1_{A_N}\| + \|S1_{A_N}\| \leq \beta^k + 1/N.\n\]

Thus for \(k\) sufficiently large we have for each \(1 \leq n \leq N\), \(\|V^k1_{A_n}\| \leq 2/N\) but \(N\) is arbitrary, hence \(\lim_{k \to \infty} \|V^k1_{A_n}\| = 0, \forall n\).

**Theorem.** Let \(X = X_0 \cup X_1\) be as in Lemma 2.3. Then there exists an operator \(U \in \Pi\) such that \(X_1\) and \(X_0\) are the conservative and dissipative parts, respectively, for \(U\).

**Proof.** Let \(V\) be as in Lemma 2.5. Define the sequence of integers \(\{n_k\}\) inductively:

\[
n_1 = 1, \quad n_{k+1} = n_k + 1, \quad \|V^j1_{A_{n_k+1}}\|_\infty \leq \frac{1}{n_k + 1}, \quad \forall j \geq k + 1,\n\]

\[
= n_k, \quad \text{otherwise.}
\]

Clearly \(n_k \to \infty\), and \(\lim_{k \to \infty} \|V^k1_{A_{n_k}}\| = 0\), where \(\{A_n\}\) is the sequence of Lemma 2.3, part (iii). Let \(\{c_k\}\) be the sequence of Lemma 3 of [1] such that \(\sum_{k=0}^{\infty} c_k \|V^k1_{A_{n_k}}\|_\infty < \infty\). It is obvious that \(\sum_{k=0}^{\infty} c_k \|V^k1_{A_{n_k}}\|_\infty < \infty\) for each \(n\).
Now let \( \{a_k\} \) be the sequence associated with \( \{c_k\} \) as in Lemma 2 of [1] and let \( U = \sum_{k=0}^{\infty} a_k V^k \), then \( U \in \Pi \) and the proof of Theorem 1 of [1] shows that \( \sum_{k=0}^{\infty} U^k i A_n \in L_\alpha \) for each \( n \), hence \( X_0 = \bigcup_{n=1}^{\infty} A_n \) is contained in the dissipative part with respect to \( U \). On the other hand if \( \mu \) is a finite invariant measure for \( \Pi \), supported on \( X_1 \), then \( \mu U = \mu \) and \( X_1 \) is contained in the conservative part with respect to \( U \).

REFERENCES


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