ON A CONJECTURE OF A. J. HOFFMAN. II
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ABSTRACT. It is proved that certain incidence relations of hyperplanes and closed convex sets in a d-polytope can be preserved while replacing these sets by suitable polytopal subsets.

The purpose of this paper is to prove

THEOREM 1. If P is a d-polytope in $E^d$ and $C_1, \ldots, C_k$ are closed convex subsets of P, such that every hyperplane that meets P meets $\bigcup_{i=1}^{k} C_i$, then there exist polytopes $D_1, \ldots, D_k$ with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every hyperplane that meets P meets $\bigcup_{i=1}^{k} D_i$.

This settles all the cases $(d, d-1, k)$, for all $d \geq 2$ and $k \geq 1$, of the following conjecture due to A. J. Hoffman [3]:

Conjecture $(d, t, k)$. If P is a d-polytope in $E^d$, $d \geq 1$; $t \geq 0$ and $k \geq 1$ are integers, $C_1, \ldots, C_k$ are closed convex subsets of P such that every (affine) t-flat that meets P meets $\bigcup_{i=1}^{k} C_i$, then there are polytopes $D_1, \ldots, D_k$ with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every t-flat that meets P meets $\bigcup_{i=1}^{k} D_i$.

A. J. Hoffman proved [3] conjecture $(d, 0, k)$, for all $d \geq 1$ and $k \geq 1$; in these cases the t-flats are points and $C_1, \ldots, C_k$ cover P.

It follows quite elementarily that conjecture $(d, t, 1)$ is true for all $d \geq 1$ and $t \geq 0$, since in this case $C_1 = P$ (see Remark 1, here). W. R. Hare, Jr. and C. R. Smith proved [2] that conjecture $(d, t, 2)$ is true for all $d \geq 1$ and $t \geq 0$. We have previously shown [4] that conjecture $(d, d-2, k)$ is false for all $d \geq 3$ and $k \geq 4$, while here it is shown that conjecture $(d, d-1, k)$ is true for all $d \geq 2$ and $k \geq 1$. Conjecture $(3, 1, 3)$ is true (see Remark 4).

DEFINITIONS. A polytope $P$ is the convex hull of a finite set of points in the Euclidean d-dimensional space $E^d$; a d-polytope in $E^d$ is a polytope with nonempty interior; Vert $P$ denotes here the set of vertices of a polytope $P$; if $A \subseteq E^d$, conv $A$ denotes the convex hull of $A$.

An (affine) t-flat in $E^d$ is a translate of a t-dimensional subspace of $E^d$, and a hyperplane is a $(d-1)$-flat. If $H$ is a hyperplane in $E^d$, then $H_+$ and $H_-$ ($H_+$ and $H_-$) denote the two closed (open, respectively) half-spaces of $E^d$, determined by $H$.

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A hyperplane $H$ supports a compact set $X$ in $E^d$ if $X \subset H_+$ and $H \cap X \neq \emptyset$. A hyperplane $H$ separates (strictly separates) $A$ and $B$ if $A \subset H_+$ and $B \subset H_-(A \subset H_+ \text{ and } B \subset H_-)$, respectively. For additional definitions and information the reader is referred to [1].

Remark 1. If $P, C_1, \ldots, C_k$ are as given in conjecture $(d, t, k)$, for some $d \geq 1$, $k \geq 1$, and $0 \leq t \leq d-1$, then $Vert P \subseteq \bigcup_{i=1}^{k} C_i$. To establish this, let $v \in Vert P$; there exists a hyperplane $H_v$ of $P$ such that $H_v \cap P = \{v\}$. $H_v$ clearly contains a $t$-flat $F_v$ through $v$. Since $F_v$ is a $t$-flat that meets $P$ (at $v$), it follows by the assumptions that $F_v$ meets $\bigcup_{i=1}^{k} C_i$, i.e., $F_v \cap (\bigcup_{i=1}^{k} C_i) \neq \emptyset$, and since $\bigcup_{i=1}^{k} C_i \subset P$, $F_v \cap (\bigcup_{i=1}^{k} C_i) \subset F_v \cap P = \{v\}$; hence $v \in \bigcup_{i=1}^{k} C_i$ and therefore $Vert F_v \subseteq \bigcup_{i=1}^{k} C_i$.

In case $k = 1$, it follows that $Vert P \subseteq C_1$, and therefore $C_1 = P$, since $C_1$ is a convex set contained in $P$. As a result, conjecture $(d, t, 1)$ is (trivially) true for all $d \geq 1$ and $0 \leq t \leq d-1$.

We need the following:

Lemma 1. If $C$ is a convex set in $E^d$, $x \in E^d$, and $A$ and $B$ are such that $C = \text{conv} (A \cup B)$, then every hyperplane that meets $C$ meets $\text{conv} (A \cup x) \cup \text{conv} (B \cup x)$.

Proof. Let $Y$ be defined by $Y = \text{conv} (A \cup x) \cup \text{conv} (B \cup x)$; every pair of points $y_1$ and $y_2$ of $Y$ are connected by the polyhedral path $y_1x \cup xy_2$ which lies entirely in $Y$.

Let $H$ be an arbitrary hyperplane such that $H \cap C \neq \emptyset$. If $A \subset H_+$ and $B \subset H_+$, then $\text{conv} (A \cup B) \subset H_+$, hence $C \subset H_+$, but this implies $C \cap H = \emptyset$ — a contradiction; therefore $A \cup B \notin H_+$, and similarly $A \cup B \notin H_-$.

Therefore $A \cup B$ contains a point $y_1$ in $H_+$ and a point $y_2$ in $H_-$; $y_1, y_2 \in Y$ since $Y \supseteq A \cup B$. The polyhedral path in $Y$ that connects $y_1$ to $y_2$ clearly meets $H$, hence $Y \cap H \neq \emptyset$ and the proof is complete.

Corollary 1. If $P$ is a polytope in $E^d$, $x \in E^d$, and $Vert P = A \cup B$, then every hyperplane that meets $P$ meets $\text{conv}(A \cup x) \cup \text{conv}(B \cup x)$.

Proof. In this case $P = \text{conv}(Vert P) = \text{conv}(A \cup B)$, and Lemma 1 is applicable (with $P = C$).

Since the replacement of $k$ compact convex sets by polytopes can be done one at a time, we state and prove the following:

Theorem 2. If $P$ is a $d$-polytope in $E^d$, $C_1, \ldots, C_k$ are closed convex subsets of $P$ such that every hyperplane that meets $P$ meets $\bigcup_{i=1}^{k} C_i$, then there exists a polytope $D_1$ in $C_i$ such that every hyperplane that meets $P$ meets $D_1 \cup \bigcup_{i=2}^{k} C_i$.

Theorem 3. If $P$ is a $d$-polytope in $E^d$, $C_1, \ldots, C_k$ are closed convex subsets of $P$, $C_i \cap C_j = \emptyset$ for all $1 \leq i < j \leq k$ and every hyperplane that meets
If \( P \) meets \( \bigcup_{i=1}^{k} C_i \), then there exists a polytope \( D_1 \) in \( C_1 \) such that every hyperplane that meets \( P \) meets \( D_1 \cup \bigcup_{i=2}^{k} C_i \).

Clearly Theorems 1 and 2 are equivalent; our proof of Theorem 1 uses the following.

**Claim 1.** Theorem 3 implies Theorem 2.

**Proof of Claim 1.** Assuming Theorem 3 is true, we prove Theorem 2 by induction on \( k \). In Case \( k=1 \), \( C_1=P \) (by Remark 1), and one chooses \( D_1=C_1 \).

Assume inductively that the assertion is true for \( k-1 \), \( k \geq 2 \), and let \( P, C_1, \ldots, C_k \) be given as described in the statement of Theorem 2. Since the inductive assumption takes care of the cases in which \( C_i=\emptyset \) for some \( 1 \leq i \leq k \), we assume that \( C_i \neq \emptyset \) for all \( 1 \leq i \leq k \). If \( C_i \cap C_j=\emptyset \) for all \( 1 \leq i<j \leq k \), then the existence of \( D_1 \) with the required property is guaranteed by Theorem 3, which is assumed to hold. Otherwise, let \( m \) and \( n \) be such that \( C_m \cap C_n \neq \emptyset \) and \( 1 \leq m<n \leq k \), and let \( x \in C_m \cap C_n \).

Define \( C_1^*, \ldots, C_{k-1}^* \) by

\[
C_i^* = \begin{cases} 
\text{conv}(C_m \cup C_n) & \text{if } i=m, \\
C_i & \text{if } i < m \text{ or } m < i < n, \\
C_{i+1} & \text{if } i \geq n.
\end{cases}
\]

Clearly, \( P \) together with \( C_1^*, \ldots, C_{k-1}^* \) satisfy all the conditions of Theorem 2, hence the inductive assumption implies that there exists a polytope \( D_1^* \) in \( C_1^* \) such that every hyperplane that meets \( P \) meets \( D_1^* \cup \bigcup_{i=2}^{k} C_i^* \).

**Case 1.** \( m \neq 1 \). Choose \( D_1=D_1^* \). If \( H \) is a hyperplane that meets \( P \), then \( H \) meets \( D_1 \cup \bigcup_{i=2}^{k} C_i^* \), i.e. \( H \) meets \( D_1 \cup \bigcup_{i \geq 2; i \neq m} C_i \cup \text{conv}(C_m \cup C_n) \).

If \( H \) meets \( D_1 \cup \bigcup_{i \geq 2; i \neq m} C_i \), then clearly \( H \) meets \( D_1 \cup \bigcup_{i=2}^{k} C_i \), as required. If \( H \) meets \( \text{conv}(C_m \cup C_n) \), then by Lemma 1, \( H \) meets \( \text{conv}(C_m \cup y) \cup \text{conv}(C_n \cup y) \) for every \( y \in E^d \), hence in particular \( H \) meets \( \text{conv}(C_m \cup x) \cup \text{conv}(C_n \cup x) \) (where \( x \in C_m \cap C_n \)); this means that \( H \) meets \( C_m \cap C_n \), because \( C_m \) and \( C_n \) are convex sets. Therefore \( H \) meets \( D_1 \cup \bigcup_{i=2}^{k} C_i \), as required.

**Case 2.** \( m=1 \). In this case \( D_1^* \subset C_1^*=\text{conv}(C_1 \cup C_n) \). Every vertex of \( D_1^* \) is a finite convex combination of points of \( C_1 \) and points of \( C_n \), therefore there exists a polytope \( D_1^{**} \) in \( \text{conv}(C_1 \cup C_n) \) such that \( D_1^{**} \supset D_1^* \) and \( \text{Vert}(D_1^{**}) \subset C_1 \cup C_n \).

Define \( D_1 \) by \( D_1=\text{conv}[x \cup (\text{Vert } D_1^{**} \cap C_1)] \), where \( x \in C_1 \cap C_n \). Let \( H \) be an arbitrary hyperplane that meets \( P \); \( H \) meets \( D_1^* \cup \bigcup_{i=2; i \neq m}^{k-1} C_i= D_1^* \cup \bigcup_{i \geq 2; i \neq m} C_i \), and since \( D_1^{**} \supset D_1^* \), \( H \) meets \( D_1^{**} \cup \bigcup_{i \geq 2; i \neq m} C_i \). If \( H \) meets \( \bigcup_{i \geq 2; i \neq m} C_i \), then clearly \( H \) meets \( D_1 \cup \bigcup_{i=2; i \neq m}^{k-1} C_i \), as required. If \( H \) meets \( D_1^{**} \), then by Corollary 1, \( H \) meets \( \text{conv}[x \cup (\text{Vert } D_1^{**} \cap C_1)] \cup \text{conv}[x \cup (\text{Vert } D_1^{**} \cap C_n)] \); the first set in this union is \( D_1 \), by the definition
of \( D_1 \), and therefore if \( H \) meets \( D_1 \) it clearly meets \( D_1 \cup \bigcup_{i=2}^{k} C_i \), as required. If, however, \( H \) meets \( \text{conv}[x \cup \text{Vert}(D^*_1 \cap C_n)] \), then \( H \) meets \( C_n \), since \( x \in C_1 \cap C_n \subseteq C_n \) and \( C_n \) is a convex set; therefore \( H \) meets \( D_1 \cup \bigcup_{i=2}^{k} C_i \), and Claim 1 has been established.

**Proof of Theorem 3.** By induction on \( k \), starting with the case \( k=1 \) being true by Remark 1. Assume inductively that the assertion is true for \( k-1, k \geq 2 \), and let \( P, C_1, \ldots, C_k \) be given as described in the statement of Theorem 3 (assume as before that \( C_i \neq \emptyset \) for all \( 1 \leq i \leq k \)).

Suppose first that for some \( j \), \( 1 \leq j \leq k \), every hyperplane that meets \( P \) meets \( \bigcup_{i \neq j} C_i \). If \( j \neq 1 \), then by the inductive assumption applied to \( P, C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_k \), there exists a polytope \( D_1 \) in \( C_1 \), such that every hyperplane that meets \( P \) meets \( D_1 \cup \bigcup_{i \neq j} C_i \); therefore every hyperplane that meets \( P \) meets \( D_1 \cup \bigcup_{i=2}^{k} C_i \), as required. If \( j = 1 \), then any choice for a \( D_1 \) in \( C_1 \) will do.

If however there is no such \( j \), then there are hyperplanes that meet \( C_1 \) and do not meet \( \bigcup_{i=2}^{k} C_i \).

For every hyperplane \( H \) with \( \cap C_1 \neq \emptyset \) and \( \cap (\bigcup_{i \neq 1} C_i) = \emptyset \) define \( K_1 \) and \( K_2 \) by

\[
K_1 = \text{conv} \bigcup \{ C_i \mid C_i \subseteq H^+ \} \quad \text{and} \quad K_2 = \text{conv} \bigcup \{ C \mid C \subseteq H_- \}.
\]

Clearly \( K_1 \subseteq H^+ \) and \( K_2 \subseteq H_- \), hence \( K_1 \cap K_2 = \emptyset \); not both of \( K_1 \) and \( K_2 \) are empty since \( \cap (\bigcup_{i \neq 1} C_i) = \emptyset \), \( k \geq 2 \) and \( C_i \neq \emptyset \) for all \( 1 \leq i \leq k \).

**Claim 2.** If \( K_1 \neq \emptyset \) and \( K_2 \neq \emptyset \), then \( C \) contains a segment \( L \) such that if a hyperplane \( F \) separates \( K_1 \) and \( K_2 \) then \( F \cap L = \emptyset \).

**Proof.** If \( K_1 \cap \text{conv}(K_2 \cup C_1) = \emptyset \), then \( K_1 \) and \( \text{conv}(K_2 \cup C_1) \) can be strictly separated by a hyperplane \( F_0 \); hence \( F_0 \cap \bigcup_{i=1}^{k} C_i = \emptyset \); however \( K_1 \neq \emptyset \), \( \text{conv}(K_2 \cup C_1) \neq \emptyset \) and \( P \) being convex imply that \( F_0 \cap P \neq \emptyset \), which contradicts the assumption on \( P, C_1, \ldots, C_k \). Therefore \( K_1 \cap \text{conv}(K_2 \cup C_1) \neq \emptyset \) and similarly \( K_2 \cap \text{conv}(K_1 \cup C_1) \neq \emptyset \).

Take \( x_1 \in K_1 \cap \text{conv}(K_2 \cup C_1) \); then there are points \( y_1 \in K_2 \) and \( z_1 \in C_1 \) such that \( x_1 = \lambda y_1 + (1-\lambda)z_1 \), for some \( 0 \leq \lambda \leq 1 \); similarly take \( y_2 \in K_2 \cap \text{conv}(K_1 \cup C_1) \), then there are points \( x_2 \in K_1 \) and \( z_2 \in C_1 \) such that \( y_2 = \mu x_2 + (1-\mu)z_2 \), for some \( 0 \leq \mu \leq 1 \) (see Figure 1).

The promised segment \( L \) in \( C_1 \) is taken as the segment \( [z_1 z_2] \). Suppose a hyperplane \( F \) separates \( K_1 \) and \( K_2 \), so that say \( K_1 \subseteq F_+ \) and \( K_2 \subseteq F_- \). Clearly \( x_1, x_2 \in K_1 \subseteq F_+ \) and \( y_1, y_2 \in K_2 \subseteq F_- \). Since \( x_1 = \lambda y_1 + (1-\lambda)z_1 \) and \( 0 \leq \lambda \leq 1 \) it follows that \( z_1 \in F_+ \), and similarly \( y_2 = \mu x_2 + (1-\mu)z_2 \) and \( 0 \leq \mu \leq 1 \) imply \( z_2 \in F_- \). As a result \( F \) meets the segment \( [z_1 z_2] = L \), and Claim 2 has been established.

**Claim 3.** If \( K_1 = \emptyset \), then \( C_1 \) contains a segment \( L \) such that if a hyperplane \( F \) meets \( C_1 \) and \( \bigcup_{i \neq 1} C_i \subseteq F_+ \) (or \( \bigcup_{i \neq 1} C_i \subseteq F_- \)), then

\[
F \cap \text{conv}(L \cup (\text{Ext} \cap C_1)) \neq \emptyset.
\]
Figure 1

Proof. \( C_1 \cap \text{conv}(\bigcup_{i \neq 1} C_i) \neq \emptyset \) since otherwise \( C_1 \) and \( \text{conv}(\bigcup_{i \neq 1} C_i) \) are strictly separated by a hyperplane \( F_0 \), hence \( F_0 \cap (\bigcup_{i=1}^k C_i) = \emptyset \); since \( P \) is convex it follows that \( F_0 \cap P \neq \emptyset \) which contradicts the assumption on \( P, C_1, \ldots, C_k \). Let \( x \in C_1 \cap \text{conv}(\bigcup_{i \neq 1} C_i) \), and take for the segment \( L \) any segment in \( C_1 \) containing \( x \) (in fact \( L = \{x\} \) is as good).

Suppose \( F \) is a hyperplane that meets \( C_1 \) and \( \bigcup_{i \neq 1} C_i \subset \hat{F}_+ \); therefore \( \text{conv}(\bigcup_{i \neq 1} C_i) \subset \hat{F}_+ \) and hence \( x \in \hat{F}_+ \). Next \( F \cap C_1 \neq \emptyset \), hence \( F_- \cap C_1 \neq \emptyset \) and therefore \( F_- \cap P \neq \emptyset \); hence \( F_- \cap \text{Ext} P \neq \emptyset \). Moreover \( F_- \cap \text{Ext} P \subset C_1 \), because \( \text{Vert} P \subset \bigcup_{i=1}^k C_i \) (see Remark 1), and hence \( F_- \cap \text{Ext} P \subset F_- \cap \bigcup_{i=1}^{k-1} C_i = F_- \cap C_1 \subset C_1 \).

We conclude that \( x \in \hat{F}_+ \) and \( C_1 \) contains a vertex \( y \) of \( P \) with \( y \in F_- \); therefore \( F \cap \text{conv}[L \cup (\text{Ext} P \cap C_1)] \neq \emptyset \).

Claim 3 has been established.

Let \( L_1, \ldots, L_r \) be a collection of segments in \( C_1 \), each one obtained by applying Claims 2 and 3 to each and every different division \( \{2, \ldots, k\} = I \cup J \) with \( I \cap J = \emptyset \), for which there exists a hyperplane \( H \) with \( H \cap C_1 \neq \emptyset \), \( \hat{H}_+ \supset \bigcup_{i \in I} C_i \) and \( \hat{H}_- \supset \bigcup_{j \in J} C_i \).

Define \( D_1 \) by \( D_1 = \text{conv}([L_1 \cup (\text{Ext} P \cap C_1)]) \). To show that \( D_1 \) has the required property as claimed in Theorem 3, suppose a hyperplane \( H \) meets \( P \). If \( P \cap (\bigcup_{i \neq 1} C_i) \neq \emptyset \), then clearly \( P \cap (D_1 \cup (\bigcup_{i \neq 1} C_i)) \neq \emptyset \). Otherwise \( P \cap (\bigcup_{i \neq 1} C_i) = \emptyset \), and since \( H \cap P \neq \emptyset \), it follows that \( H \cap C_1 \neq \emptyset \). By Claim 2 or 3, \( C_1 \) contains the segment \( L_j \) for the appropriate \( j, 1 \leq j \leq r \), such that \( H \cap \text{conv}[L_j \cup (\text{Ext} P \cap C_1)] \neq \emptyset \), hence \( H \cap D_1 \neq \emptyset \) as needed. \( D_1 \) is clearly a polytope in \( C_1 \).

This completes the proof of Theorem 3.
The proof of Theorem 1 follows now easily from the proof of Theorem 3, which implies Theorem 2 by Claim 1, and the equivalence of Theorems 2 and 1.

Remark 2. Shortly before proving conjecture \((d, d-1, k)\) for all \(d \geq 2\) and \(k \geq 1\), we established conjecture \((2, 1, k)\) for all \(k \geq 1\), using the following:

Claim 4. If \(C_1\) and \(C_2\) are disjoint compact convex sets in \(E^2\), then they have at most four (4) common supporting lines.

Claim 5. If \(C_1\) and \(C_2\) are disjoint compact convex sets in \(E^d, d \geq 2\), \(\{H_i | i \in I\}\) the collection of all the common supporting hyperplanes to \(C_1\) and \(C_2\), \(x_i \in H_i \cap C_1\) and \(y_i \in H_i \cap C_2\) for all \(i \in I\), then every hyperplane that meets both \(C_1\) and \(C_2\) meets \(\operatorname{conv}(x_i | i \in I) \cup \operatorname{conv}(y_i | i \in I)\).

Both Claims 4 and 5 in the case \(d = 2\) imply the following: “If \(C_1\) and \(C_2\) are disjoint compact convex sets in \(E^2\), then there exist convex quadrangles \(D_1\) and \(D_2\), \(x_i \in D_i \subseteq C_i\) for \(i = 1, 2\), such that every hyperplane that meets both \(C_1\) and \(C_2\) meets \(D_1 \cup D_2\).” Unfortunately, the index set \(I\) in Claim 5 is infinite for all \(d \geq 3\), and there is no valid analogue of the last theorem for \(E^d, d \geq 3\), with “convex quadrangle” replaced by “polytopes” (take, for example, two disjoint balls).

Remark 3. Lemma 1 can be extended as follows:

Lemma 2. If \(C\) is a convex set in \(E^d, x \in E^d\), and \(\{A_i | i \in I\}\) are such that \(C = \operatorname{conv} \bigcup \{A_i | i \in I\}\), then every hyperplane that meets \(C\) meets \(\bigcup_{i \in I} \operatorname{conv}(x \cup A_i)\).

The proof is similar to the proof of Lemma 1, hence it is omitted.

Corollary 2. If \(P\) is a polytope in \(E^d, x \in E^d\), and \(\{v_1, \ldots, v_n\} = \operatorname{Vert} P\), then every hyperplane that meets \(P\) meets \(\bigcup_{i=1}^{n} [x, v_i]\).

Let a graph (= finite 1-dimensional simplicial complex) be called starshape if it has exactly \(n+1\) vertices, one of valence \(n\) and \(n\) of valence 1, \(n \geq 1\).

Corollary 3. If \(P\) is a \(d\)-polytope in \(E^d\), \(C_1, \ldots, C_k\) are closed convex subsets of \(P\), such that every hyperplane that meets \(P\) meets \(\bigcup_{i=1}^{k} C_i\), then there exist starshapes \(G_1, \ldots, G_k\) with \(G_i \subseteq C_i\) for all \(1 \leq i \leq k\), such that every hyperplane that meets \(P\) meets \(\bigcup_{i=1}^{k} G_i\).

Proof. There exist, by Theorem 1, polytopes \(D_1, \ldots, D_k\) with \(D_i \subseteq C_i\) for all \(1 \leq i \leq k\), such that every hyperplane that meets \(P\) meets \(\bigcup_{i=1}^{k} D_i\); let \(x_i \in D_i\), and define \(G_i\) by \(G_i = \bigcup \{[x, v_i] | v_i \in \operatorname{Vert} D_i\}\), for all \(1 \leq i \leq k\). \(G_i\) is a starshape, for all \(1 \leq i \leq k\), and every hyperplane that meets \(\bigcup_{i=1}^{k} D_i\) meets \(\bigcup_{i=1}^{k} G_i\), by Corollary 2.
Remark 4. Conjecture (3, 1, 3) has been recently established by the author of this paper, using some ideas of [2]; the proof will appear.

Remark 5. As stated in [3], it was M. O. Rabin who first proposed conjecture \((d, 0, k)\), for all \(d \geq 1\) and \(k \geq 1\).

REFERENCES


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