ON CERTAIN FIBERINGS OF $M^2 \times S^1$

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Abstract. Using a theorem of Stallings it is shown that the product of $S^1$ and a surface of genus $g > 1$ admits for every integer $n \geq 0$ a fibering over $S^1$ with a surface of genus $n(g-1)+g$ as fiber. Conversely, these are all possible such fibrations (up to equivalence). Let $N$ be a Seifert fiber space which is locally trivial fibred over $S^1$ with fiber a surface. It is shown that any two such fiberings of $N$ over $S^1$ are equivalent if the fibers are homeomorphic.

In [8] and [1] it is shown that the 3-manifold $M = F \times S^1$, where $F$ is an orientable closed surface of genus $g > 1$, admits for every number $n \geq 0$ a fibering over $S^1$ with a surface $T_n$ of genus $n(g-1)+g$ as fiber. In this note we show that this result follows immediately from Stallings' theorem [7] (this applies also if $F$ is bounded or nonorientable). It is shown that these are all possible fibrations of $M$ over $S^1$ with fiber a surface and this is generalized to Seifert fiber spaces.

1. Let $F$ be an orientable surface of genus $g > 1$ and $m$ boundary components, let $M = F \times S^1$, $\mathcal{G} = \pi_1(M)$,

$\mathcal{G} = \{a_1, b_1, \ldots, a_g, b_g, s_1, \ldots, s_m, h : s_1 \cdots s_m[a_1, b_1] \cdots [a_g, b_g] = 1, [a_i, h] = [b_i, h] = [s_k, h] = 1 \ (i = 1, \ldots, g; k = 1, \ldots, m)\}.$

Let $Z$ be represented by the group of integers and construct an epimorphism $\phi: \mathcal{G} \rightarrow Z$ as follows

$\phi(a_1) = 1,$

$\phi(a_i) = \phi(b_i) = 0 \quad (i = 2, \ldots, g; j = 1, \ldots, g),$  

$\phi(h) = n > 0,$

$\phi(s_k) = \gamma_k \quad (k = 1, \ldots, m).$

($\gamma_k$ are arbitrary integers, subject to the condition $\gamma_1 + \cdots + \gamma_m = 0.$)
(a) If $F$ is closed (i.e. $m=0$), computing $\mathfrak{N}_n = \ker \phi$ using the Reidemeister-Schreier method, we obtain

$$\mathfrak{N}_n = \left\{ a_{i,k}, b_{j,k}, h_k; a_{i,k}h_k a_{i,k+n}^{-1} = 1, b_{j,k}h_k b_{j,k+n}^{-1} = 1, h_{k+1} h_k^{-1} = 1, b_{1,k+1} b_{1,k}^{-1} \prod_{i=2}^{g} [a_{i,k}, b_{i,k}] = 1 \right\}$$

Here $a_{i,k} = a_1^k a_1^{-k}$, $b_{j,k} = b_j b_1^{-k}$, $h_k = h_k h_1^{-i(k+n)}$. This is equivalent to

$$\mathfrak{N}_n = \left\{ h_0, b_{1,1}, a_{i,1}, b_{j,1}, \cdots, a_{i,n}, b_{j,n}; [h_0^{-1}, b_{1,1}] \prod_{j=2}^{g} [a_{j,1}, b_{j,1}] \times \prod_{j=2}^{g} [a_{j,2}, b_{j,2}] \cdots \prod_{j=2}^{g} [a_{j,n}, b_{j,n}] = 1 \right\}$$

which is the fundamental group of an orientable closed surface of genus $n(g-1)+1$. Thus the theorem in the introduction follows by applying Stallings' theorem [7].

(b) If $\partial M \neq \emptyset$ (i.e. $m > 0$) we obtain, for $\mathfrak{N}_n = \ker \phi$,

$$\mathfrak{N}_n = \left\{ a_{i,k}, b_{j,k}, s_{i,k}, h a_1^{-s}(i = 2, \cdots, g; j = 1, \cdots, g; k = 0, \cdots, n-1; l = 1, \cdots, m-1) \right\}$$

(where $s_{i,k} = a_1^k s_i a_1^{-k}$),

a free group of rank $n(2g+m-2)+1$. By Stallings' theorem $M$ fibers over $S^1$ with fiber a surface $T_n$ with $\pi_1(T_n) = \mathfrak{N}_n$. $M$ is a (trivial) Seifert fiber space with orbit surface $F$. $T_n$ is a branched covering of $F$ (see the proposition, §3). Since $M$ has no singular fibers this covering is without branch points. Thus if $g'$ denotes the genus and $m'$ the number of boundary components of $T_n$ and if the covering $T_n \rightarrow F$ is $\eta$-sheeted, we have for the Euler characteristics

$$2g' + m' - 2 = \eta(2g + m - 2) = n(2g + m - 2).$$

Thus: For every natural number $n$ there exists a surface $T_n$ which is an $n$-sheeted covering of $F$ and such that $M$ admits a fibering over $S^1$ with fiber $T_n$.

(c) The same method carries over to the nonorientable case.

2. The fiberings of §1 are all possible fiberings of $M$ over $S^1$ with fiber a surface. This can be seen as follows:

Let $\phi: \mathbb{G} \rightarrow \mathbb{Z}$ be any epimorphism.
Let
\[ \phi(a_i) = \alpha_i \quad (i = 1, \ldots, g), \]
\[ \phi(b_i) = \beta_i \quad (i = 1, \ldots, g), \]
\[ \phi(s_k) = \gamma_k \quad (k = 1, \ldots, m), \]
\[ \phi(h) = n. \]

Let \( \text{g.c.d.}(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) = d \). Since \( \phi \) is an epimorphism, we have \( \text{g.c.d.}(d, \gamma_1, \ldots, \gamma_m, n) = 1. \)

The assertion follows from the following:

**Lemma.** Let \( \phi : \mathfrak{G} \to \mathbb{Z} \) be any epimorphism and let \( x \) be any one of the generators \( a_1, b_1, \ldots, a_g, b_g \). Then there exists an automorphism \( \mu : \mathfrak{G} \to \mathfrak{G} \) which is induced by a homeomorphism of \( M \), such that \( \phi \cdot \mu(x) = \text{g.c.d.}(d, n) \) and \( \phi \cdot \mu(y) = 0 \), where \( y \in \{a_1, b_1, \ldots, a_g, b_g\} \setminus \{x\} \). If \( F \) is not a torus, we may assume \( \phi(h) > 0 \).

**Proof.** \( \mu \) is a composition of the following automorphisms (we write down the generators which are not kept fixed).

\[ \mu_1^{(i)}(a_i) = a_i b_i^k \quad (k \in \mathbb{Z}) \quad (i = 1, \ldots, g), \]
\[ \mu_2^{(i)}(b_i) = b_i a_i^l \quad (l \in \mathbb{Z}) \quad (i = 1, \ldots, g), \]
\[ \mu_3(a_1) = a_1 a_2 b_2^{-1}, \]
\[ \mu_3(b_1) = b_2 a_2^{-1} b_1 a_2 b_2^{-1}, \]
\[ \mu_3(a_2) = b_2 a_2^{-1} b_1 a_2 b_2^{-1} b_1^{-1} a_2 a_2^{-1} b_1^{-1} a_2 b_2^{-1}, \]
\[ \mu_3(b_2) = b_2 b_2 a_2^{-1} b_1^{-1} a_2^{-1} b_2^{-1}, \]
\[ \mu_4(a_1) = a_1 a_2^{-1} b_2^{-1}, \]
\[ \mu_4(b_1) = b_2 a_2 b_1 a_2^{-1} b_2^{-1}, \]
\[ \mu_4(a_2) = b_2 a_2 b_1 a_2^{-1} b_2^{-1} b_1^{-1} a_2^{-1} b_1^{-1} a_2 b_2^{-1}, \]
\[ \mu_4(b_2) = b_2 b_2 a_2^{-1} a_2^{-1} b_2^{-1}, \]
\[ \mu_5^{(i)}(a_i) = a_{i+1}, \]
\[ \mu_5^{(i)}(b_i) = b_{i+1}, \]
\[ \mu_5^{(a_{i+1})} = [a_{i+1}, b_{i+1}]^{-1} a_{i}[a_{i+1}, b_{i+1}], \]
\[ \mu_5^{(b_{i+1})} = [a_{i+1}, b_{i+1}]^{-1} b_{i}[a_{i+1}, b_{i+1}] \quad (i \text{ taken mod } g), \]
\[ \mu_5(a_i) = a_i h_{i+1}, \]
\[ \mu_5(h) = h^{-1}. \]

It is not difficult to see that these are automorphisms and furthermore that they are induced by homeomorphisms of \( M \), since they leave the
peripheral system of $G$ fixed (see [2]). These automorphisms were sug-
ggested by the paper of J. Nielsen [3].

Let

$$A = \begin{pmatrix}
\alpha_1 & \beta_1 \\
\vdots & \vdots \\
\alpha_g & \beta_g \\
\end{pmatrix}.$$ 

The automorphisms $\mu_1$ and $\mu_2$ change the map $\phi$ as follows:

$$(\mu_1) \quad \phi(a_i) \to \phi(a_i) + k\phi(b_i),$$

$$(\mu_2) \quad \phi(b_i) \to \phi(b_i) + l\phi(a_i).$$

Using the Euclidean algorithm and $(\mu_1), (\mu_2)$, we transform $A$ into

$$A' = \begin{pmatrix}
d_1 & 0 \\
\vdots & \vdots \\
d_g & 0 \\
\end{pmatrix}, \quad \text{where } d_i = (\alpha_i, \beta_i).$$

Similarly, using $\mu_3, \mu_4, \mu_5, \mu_6$ we change $A'$ into

$$\begin{pmatrix}
0, & \gcd(d, n) \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
\end{pmatrix}.$$ 

The last statement of the lemma follows by considering $\mu_1$ and observing
that $(\ker \phi) \cap Z(h) = 1$, where $Z(h)$ is the cyclic subgroup of $G$ generated
by $h$ (see [5, proof of Satz 7]).

3. A comparison of Nielsen's and Seifert's invariants. In this section
we show how for Seifert fiber spaces that are fibered over $S^1$ the fiber is
a branched covering of the Seifert (orbit) surface. This will be used (in
the next section) to show the uniqueness of the fibration as mentioned in
the introduction.

Let $\phi: F \to F$ be an orientation preserving homeomorphism of finite
order $n$ of a (compact) orientable surface $F$ of genus $g$ and $r$ boundary
components. Let $P$ be a fixed point of order $\lambda$. The orbit space of $\phi$ is a
surface $\mathcal{F}$ and $P$ covers a point $\bar{P} \in \mathcal{F}$. A simple closed curve $\mathcal{S}$ in $F$
which covers a simple closed curve $\mathcal{S}$ about $\bar{P}$, covers it $\lambda$ times. We have $m$
disjoint curves lying over $\bar{s}$, where $n = \lambda \cdot m$. Choose an orientation on $F$.

Let $\bar{Q}$ be any point on $\bar{s}$. $Q$ is covered by $\lambda$ points on $s$ lying over $\bar{Q}$. The (oriented) arc on $s$ which starts at $Q$ and covers $\bar{s}$ once ends at a certain point $\phi^{\sigma m} Q$. Note that $\text{g.c.d.}(\sigma, \lambda) = 1$. The \textit{valenz} of $P$ is defined to be the triple $(m, \lambda, \sigma)$. A multiple point is one for which $\lambda > 0$.

**Theorem (Nielsen [3]).** Let $F, F'$ be homeomorphic closed surfaces, let $\phi : F \to F$ and $\phi' : F' \to F'$ be homeomorphisms of finite order $n$. Then $\phi$ and $\phi'$ are equivalent (i.e. there exists a homeomorphism $\psi : F \to F'$ such that $\phi \psi = \psi \phi'$) iff $F$ and $F'$ have the same valenz-numbers at multiple points.

For a description of the Seifert invariants $(\mu, \nu)$ of a fibered solid torus and a 3-manifold, see [6].

Let $M$ be a Seifert fiber space which admits a fibering over $S^1$ with fiber a surface $F$ of genus $> 1$. Thus $M$ can be obtained from $F \times I$ (where $I$ denotes the unit interval $[0, 1]$) by identifying $F \times 0$ with $\phi F \times 1$, where $\phi : F \to F$ is a homeomorphism and we write $M = F \times I / \phi$. It is easy to see that $M$ is a Seifert fiber space iff $\pi_1(M)$ has nontrivial center and $\pi_2(M)$ has nontrivial center iff $\phi$ is isotopic to a homeomorphism $\phi'$ of finite order (see e.g. [9, p. 514]). Since $\phi$ and $\phi'$ determine homeomorphic 3-manifolds [2], we may assume that $\phi$ has finite order $n$. We construct a Seifert fibration of $M$ as follows: Let $P$ in $F$ be a fixed point of order $\lambda > 1$. Then $P, \phi P, \cdots, \phi^{m-1}(P)$ (where $\lambda m = n$) cover the same point $\bar{P}$ in the orbit surface $F$. Now $F \times I$ has a trivial fibering as a line bundle. Take a neighborhood $U(P)$ of $P$ which does not contain any other multiple point and such that $\phi^n(U(P)) = U(P)$. Then we have neighborhoods

$$U(P) \times I, \phi U(P) \times I, \cdots, \phi^{m-1}U(P) \times I \quad (\phi^n U(P) = U(P))$$

of $P \times I, \phi P \times I, \cdots, \phi^{m-1}(P) \times I$ in $F \times I$ and they match together to form a fibered solid torus in $M$. The fiber in $M$ which contains $P$ is composed of $m$ lines $P \times I, \cdots, \phi^{m-1}(P) \times I$ and the fiber through any point $Q$ of $U(P)$ ($Q \neq P$) is composed of $n$ lines $Q \times I, \cdots, \phi^{m-1}(Q) \times I$. Hence the fibers through $U(P)$ form a fibered neighborhood of an exceptional fiber of order $n/m = \lambda$.

Note that the orbit surface $F$ is the Seifert surface of the Seifert fibration.

Let $\bar{P} \in F$ be a multiple point of order $\lambda$, $\bar{s}$ a small simple closed curve around $\bar{P}$, $\bar{Q} \in \bar{s}$ an arbitrary point and $s$ in $F$ a closed curve which covers $\bar{s}$. On $s$ there are exactly $\lambda$ points which cover $\bar{Q}$:

$$Q, \phi^{\sigma m} Q, \cdots, \phi^{(\lambda-1)\sigma m} Q \quad \text{(exponents mod } n),$$

where $\sigma$ is the valenz. To find $\phi^{\sigma m} Q$ in this sequence, we have to find an integer $\delta$ such that $\delta \sigma \equiv 1 \pmod{\lambda}$. Now $s$ is mapped onto itself for the first time by $\phi^{\sigma}$ and $\phi^{\sigma m}$ is equivalent to a rotation of $2\pi \delta / \lambda$ of a circle. Hence the
Seifert invariants $\mu, v$ of $M$ and the valenz $(m, \lambda, \sigma)$ of the map $\phi: F \to F$ satisfy
\[
\sigma \equiv v \pmod{\mu}, \quad \text{where } \delta \sigma \equiv 1 \pmod{\lambda},
\]
\[
\lambda = \mu.
\]
Now if $M_1$ and $M_2$ are homeomorphic Seifert fiber spaces, then the corresponding Seifert surfaces are homeomorphic and $M_1$ and $M_2$ have the same numbers $\mu, v$ by the classification theorem of Seifert fiber spaces [5].

Hence we have the following:

**Proposition.** If $M_1 = F_1 \times I/\phi_1$ and $M_2 = F_2 \times I/\phi_2$ are homeomorphic and $\phi_i$ is a homeomorphism of order $n_i$ ($i = 1, 2$), then $F_1$ and $F_2$ are (branched) coverings of the same orbit surface (=Seifert surface) $F$ with the same number $t$ of branch points (on $F$) which are of the same orders $\lambda$.

4. Equivalent Stallings fibrations. Two fiberings $(M_1, p_1, S^1, F_1)$ and $(M_2, p_2, S^1, F_2)$ are equivalent iff there exists a homeomorphism $\psi: M_1 \to M_2$ with $\psi p_2 = p_1$. Let $F_i$ be a closed orientable surface of genus $g_i > 1$ ($i = 1, 2$) and let $\phi_i: F_i \to F_i$ be a homeomorphism of finite order $n_i$.

**Theorem.** Let $M_i = F_i \times I/\phi_i$ ($i = 1, 2$). Assume $F_1$ and $F_2$ are homeomorphic. Then the following are equivalent:

(a) $M_1$ is homeomorphic to $M_2$.

(b) $M_1$ is equivalent to $M_2$.

(c) $\phi_1$ is equivalent to $\phi_2$ (and is of the same order).

In particular, it follows that if $M$ is a closed Seifert fiber space which admits two fibrations over $S^1$ with fibers $F_1$ and $F_2$, then either $F_1$ is not homeomorphic to $F_2$ or the two fibrations are equivalent.

**Proof.** If $\phi_1$ and $\phi_2$ are equivalent then it is not hard to see that $M_1$ and $M_2$ are equivalent (see e.g. [2]). Thus (c) $\Rightarrow$ (b) $\Rightarrow$ (a). We show (a) $\Rightarrow$ (c):

Let $M_1$ be homeomorphic to $M_2$. $M_1$ and $M_2$ are Seifert fiber spaces and have the same Seifert surface $F$. If $t_i$ denotes the number of branch points (on $F$) of the orbit surfaces of $\phi_i$ ($i = 1, 2$) and $\lambda_i^{(j)}$ the orders of the branch points ($i = 1, 2; j = 1, \cdots, t_i$) we have (by the proposition) $t_1 = t_2 = t$ and $\lambda_i^{(1)} = \lambda_i^{(2)} = \lambda_j$ ($j = 1, \cdots, t$). Consider the branched covering $F_i \to F$ ($i = 1, 2$) and cut out a small disc $D_j$ in $F$ containing a branch point of order $\lambda_j^{(i)}$ and remove the $m_i^{(j)}$ discs in $F_i$ which cover $D_j$ (where $n_j = \lambda_j^{(i)} m_i^{(j)}$). Do this for all branch points $P_j$ ($j = 1, \cdots, t$) and get an unbranched covering $F_i' \to F'$. Clearly, if $r_i$ denotes the number of boundary components of $F_i'$, we have
\[
r_i = m_1^{(i)} + \cdots + m_t^{(i)} \quad (i = 1, 2).
\]
Using this equation together with \( n_i = \lambda_i m_j^{(i)} \) \((i=1, 2; j=1, \ldots, t)\) and comparing the Euler characteristics of \( F_i' \) and \( F' \) we get \( n_1 = n_2 \) and \( m_j^{(1)} = m_j^{(2)} \).

Now \( \phi_1 \) and \( \phi_2 \) are of the same orders and have the same valenz-numbers at the fixed points. By the Nielsen equivalence theorem \( \phi_1 \) and \( \phi_2 \) are equivalent.

**Remark.** A “mapping class” is a coset of the group of all homeomorphisms of a surface \( F \) modulo the subgroup of isotopic deformations. J. Nielsen [4, p. 24] proves that a mapping class of order \( n \) contains a homeomorphism of order \( n \). The above theorem shows that there is exactly one such homeomorphism (up to equivalence).

For let \( \phi: F \to F \) be a homeomorphism of order \( n \) and \( \psi \) be a homeomorphism of the same class. Then \( M = F \times I/\phi \approx F \times I/\psi \). If \( \psi \) has finite order, then by the theorem, \( \psi \) has order \( n \) and is equivalent to \( \phi \).

**References**