COMPARISON THEOREMS FOR NONSELFADJOINT DIFFERENTIAL EQUATIONS BASED ON INTEGRAL INEQUALITIES

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Abstract. In a variant of the classical Sturmian comparison theorem for selfadjoint Sturm-Liouville equations, A. Ju. Levin has replaced the pointwise conditions on the coefficients by an integral inequality. This theorem is generalized to apply to nonselfadjoint differential equations of the form

$$u'' + b(x)u' + c(x)u = 0.$$ 

The basic Sturmian comparison theorem deals with functions $u(x)$ and $v(x)$ satisfying

(1) $u'' + c(x)u = 0,$
(2) $v'' + \gamma(x)v = 0.$

If $\gamma(x) \geq c(x)$, then solutions of (2) oscillate more rapidly than solutions of (1). More precisely, if $u(x)$ is a nontrivial solution of (1) for which $u(x_1) = u(x_2) = 0$ ($x_1 < x_2$) and $\gamma(x) \geq c(x)$ for $x_1 \leq x \leq x_2$, then $v(x)$ has a zero in $(x_1; x_2]$. This basic result has numerous generalizations, the following of which will be relevant to this paper.

1. It is possible to replace the condition $u(x_1) = 0$ by $u'(x_1) + \sigma u(x_1) = 0$ where $\sigma$ is a constant ($-\infty \leq \sigma < \infty$) and $\sigma = -\infty$ is used to denote the condition $u(x_1) = 0$. In this case one concludes that every solution of (2) satisfying $v'(x_1) + \tau v(x_1) = 0$ with $\tau \geq \sigma$ has a zero in $(x_1; x_2]$.

2. The pointwise inequality $\gamma(x) \geq c(x)$ can be replaced by weaker integral inequalities. Such weaker conditions have been established by several authors, but of special interest to us here are the results of Levin [1] which deal with nontrivial solutions of (1) and (2) satisfying

(3) $u'(x_1) + \sigma u(x_1) = 0,$
(4) $v'(x) + \tau v(x_1) = 0,$

respectively.

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Theorem 1 (Levin). If $\sigma$ and $\tau$ are finite, $u(x_2) = 0$ for some $x_2 > x_1$, and if

$$\tau + \int_{x_1}^{x_2} \gamma(t) \, dt \geq \left| \sigma + \int_{x_1}^{x_2} c(t) \, dt \right|$$

for $x_1 \leq x \leq x_2$, then $v(x)$ has a zero in $(x_1, x_2]$.

3. Sturmian theorems have been studied for the general nonselfadjoint linear differential equation of second order. Such results have been established by Kamke [2] using the Pruefer transformation and by Swanson [3] and the author [4] as special cases of Sturmian theorems for nonselfadjoint elliptic equations. However the hypotheses required in these theorems were either pointwise inequalities or integral inequalities of a type different from Levin’s and involving the solution $u(x)$ of the differential equation which “oscillates slower.”

The purpose of this paper is to unify the three generalizations indicated above for the case of nonselfadjoint equations of the form

(5) \[ u'' - 2b(x)u' + c(x)u = 0, \]

(6) \[ v'' - 2\beta(x)v' + \gamma(x)v = 0 \]

whose coefficients are assumed to be real and continuous. (The general linear second order differential equation can always be put into such form by use of a Liouville transformation.) By means of the transformation

\[ w = -u'/u; \quad z = -v'/v; \]

the equations (5) and (6) are transformed into Riccati equations

(5') \[ w' = w^2 + 2bw + c, \]

(6') \[ z' = z^2 + 2\beta z + \gamma \]

and the initial conditions

(7) \[-u'(x_1)u(x_1) = \sigma; \quad -v'(x_1)v(x_1) = \tau\]

for (5) and (6) become initial values

(8) \[ w(x_1) = \sigma; \quad z(x_1) = \tau \]

for (5') and (6'). The differential equations (5') and (6') subject to (8) can in turn be written as integral equations

(5'') \[ w(x) = \sigma + \int_{x_1}^{x} w^2 \, dt + \int_{x_1}^{x} 2bw \, dt + \int_{x_1}^{x} c \, dt, \]

(6'') \[ z(x) = \tau + \int_{x_1}^{x} z^2 \, dt + \int_{x_1}^{x} 2\beta z \, dt + \int_{x_1}^{x} \gamma \, dt. \]
It is obvious from these equations that if \( \tau \geq \sigma \geq 0, \beta(x) \geq b(x) \geq 0, \) and

\[
\int_{z_1}^{z} \gamma(t) \, dt \geq \int_{z_1}^{z} c(t) \, dt \geq 0
\]

on an interval \([x_1, x_2]\), then \( z(x) \geq w(x) \geq 0 \) as long as \( z(x) \) can be continued on \([x_1, x_2]\). Since the singularities of \( w(x) \) and \( z(x) \) correspond to the zeros of \( u(x) \) and \( v(x) \), respectively, these observations lead to the following elementary comparison theorem for (5) and (6).

**Theorem 2.** Suppose \( u(x) \) is a nontrivial solution of (5) satisfying

\[-u'(x_1)/u(x_1) = \sigma = 0, \ u(x_2) = 0.\]

If

(i) \( \beta(x) \geq b(x) \geq 0 \) for \( x_1 \leq x \leq x_2, \)

(ii) \( \int_{x_1}^{x} \gamma(t) \, dt \geq \int_{x_1}^{x} c(t) \, dt \geq 0 \) for \( x_1 \leq x \leq x_2, \)

then every solution of (6) satisfying \(-v'(x_1)/v(x_1) \geq \sigma \) has a zero in \((x_1, x_2]\).

We shall be interested in variations of Theorem 2 which do not require the nonnegativity of \( \sigma, \tau, b(x), \) and \( \int_{x_1}^{x} c(t) \, dt \). To that end we note that the integral equations (5") and (6") can be written

\[
(5") \quad w(x) = \sigma + \int_{x_1}^{x} (w + b)^2 \, dt + \int_{x_1}^{x} (c - b^2) \, dt, \]

\[
(6") \quad z(x) = \tau + \int_{x_1}^{x} (z + \beta)^2 \, dt + \int_{x_1}^{x} (\gamma - \beta^2) \, dt. \]

This formulation shows that condition (ii) of Theorem 2 can be replaced by

\[
\int_{x_1}^{x} (\gamma - \beta^2) \, dt \geq \int_{x_1}^{x} (c - b^2) \, dt \geq 0. \]

It also allows for other results of a more general nature.

**Lemma 1.** Let \( w(x) \) and \( z(x) \) be solutions of (5") and (6"), respectively, for which \( \sigma > -\infty \) and

(i) \( \tau + \int_{x_1}^{x} (\gamma - \beta^2) \, dt > \sigma + \int_{x_1}^{x} (c - b^2) \, dt \) for \( x_1 \leq x \leq x_2, \)

(ii) \( \beta(x) \geq |b(x)| \) for \( x_1 \leq x \leq x_2, \)

Then \( z(x) > |w(x)| \) as long as \( z(x) \) can be continued on \([x_1, x_2]\).

**Proof.** From (6") we have \( z(x) \geq \tau + \int_{x_1}^{x} (\gamma - \beta^2) \, dt \) for \( x_1 \leq x \leq x_2. \) Using
(i) and (5*) this implies that

\[ z(x) > -\sigma - \int_{x_1}^{x} (c - b^2) \, dt \]
\[ > -\sigma - \int_{x_1}^{x} (c - b^2) \, dt - \int_{x_1}^{x} (w + b) \, dt > -w(x) \]

for \( x_1 \leq x \leq x_2 \). It remains to show that \( z(x) > w(x) \). To that end we assume to the contrary that there exists \( x_0 \in (x_1, x_2) \) such that \( z(x_0) \leq w(x_0) \). Then there exists an \( \tilde{x} \in (x_1, x_0) \) such that \( z(\tilde{x}) = w(\tilde{x}) \) and \( z(x) > |w(x)| \) for \( x_1 \leq x < \tilde{x} \). Using (ii) we have that

\[ z(x) + \beta(x) > |w(x)| + |b(x)| \geq |w(x) + b(x)| \quad \text{for } x_1 \leq x < \tilde{x}, \]

and consequently that \( \int_{x_1}^{x} (z + \beta)^2 \, dt > \int_{x_1}^{x} (w + b)^2 \, dt \). Using (6*), (i), and (5*) yields

\[ w(\tilde{x}) = \sigma + \int_{x_1}^{\tilde{x}} (c + b^2) \, dt + \int_{x_1}^{\tilde{x}} (w + b) \, dt \]
\[ < \tau + \int_{x_1}^{\tilde{x}} (\gamma - \beta^2) \, dt + \int_{x_1}^{\tilde{x}} (z + \beta)^2 \, dt = z(\tilde{x}) \]

which is a contradiction and establishes the lemma.

A continuity argument can now be used to establish the following.

**Lemma 2.** Let \( w(x) \) and \( z(x) \) be solutions of (5*) and (6*), respectively, for which \( \sigma > -\infty \) and

(i) \[ \tau + \int_{x_1}^{x} (\gamma - \beta^2) \, dt \geq \left| \sigma + \int_{x_1}^{x} (c - b^2) \, dt \right| \quad \text{for } x_1 \leq x \leq x_2, \]

(ii) \[ \beta(x) \geq |b(x)| \quad \text{for } x_1 \leq x \leq x_2. \]

Then \( z(x) \geq w(x) \) as long as \( z(x) \) can be continued on \([x_1, x_2]\).

As an immediate consequence of Lemma 2 we have the following generalization of Levin's Theorem 1.

**Theorem 3.** Suppose \( u(x) \) and \( v(x) \) are nontrivial solutions of (5) and (6), respectively, and that \( u(x) \neq 0 \) for \( x_1 \leq x < x_2, \ u(x_2) = 0 \). If

(i) \[ - \frac{v'(x_1)}{v(x_1)} + \int_{x_1}^{x} (\gamma - \beta^2) \, dt \geq \left| - \frac{u'(x_1)}{u(x_1)} + \int_{x_1}^{x} (c - b^2) \, dt \right| \quad \text{for } x_1 \leq x \leq x_2, \]

(ii) \[ \beta(x) \geq |b(x)| \quad \text{for } x_1 \leq x \leq x_2, \]

then \( v(x) \) has a zero in \( (x_1, x_2) \).
Theorems 2 and 3 show how to estimate zeros of the nonselfadjoint equations (6) in terms of the coefficients and the initial value of $-v'(x_1)v(x_1)$. Furthermore Theorem 2 shows that such estimates are sometimes simpler in case $\tau = -v'(x_1)v(x_1)$ is nonnegative. The following result shows how one can "shift" the initial value of $-v'(x_1)v(x_1)$ by means of a compensating shift in the coefficient of the differential equation.

**Theorem 4.** Let $v(x)$ be a solution of (6) satisfying $-v'(x_1)v(x_1) = \tau$. The first zero of $v(x)$ is the same as the first zero of $V(x)$, where $V(x)$ is a solution of

$$
V'' - 2(\beta - \tau_0)V' + (c - 2\beta\tau_0 + \tau_0^2)V = 0,
$$

(9)

$$
-V'(x_1)V(x_1) = \tau + \tau_0.
$$

**Proof.** The substitution $z(x) = -v'(x)/v(x)$ leads to

$$
z(x) = \tau + \int_{x_1}^{x} z'^2 \, dx + \int_{x_1}^{x} 2\beta z \, dx + \int_{x_1}^{x} c \, dx.
$$

Defining $Z(x) = z(x) + \tau_0$ yields

$$
Z(x) = \tau + \tau_0 + \int_{x_1}^{x} (Z - \tau_0)^2 \, dx + \int_{x_1}^{x} 2\beta(Z - \tau_0) \, dx + \int_{x_1}^{x} c \, dx
$$

$$
= \tau + \tau_0 + \int_{x_1}^{x} Z'^2 \, dx + \int_{x_1}^{x} 2(\beta - \tau_0)Z \, dx + \int_{x_1}^{x} (c - 2\beta\tau_0 + \tau_0^2) \, dx.
$$

Now the first singularity of $Z(x)$ coincides with the first singularity of $z(x)$ and therefore with the first zero of $v(x)$. But the first singularity of $Z(x)$ also coincides with the first zero of $V(x)$ satisfying (9), by the substitution $Z(x) = -V'(x)/V(x)$. This completes the proof.

The shift formula of Theorem 4 can also be applied in connection with other known comparison theorems for nonselfadjoint differential equations [4].

**Bibliography**


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