

COMPARISON THEOREMS FOR NONSELFADJOINT
DIFFERENTIAL EQUATIONS BASED ON
INTEGRAL INEQUALITIES

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ABSTRACT. In a variant of the classical Sturmian comparison theorem for selfadjoint Sturm-Liouville equations, A. Ju. Levin has replaced the pointwise conditions on the coefficients by an integral inequality. This theorem is generalized to apply to non-selfadjoint differential equations of the form

$$u'' + b(x)u' + c(x)u = 0.$$

The basic Sturmian comparison theorem deals with functions $u(x)$ and $v(x)$ satisfying

$$(1) \quad u'' + c(x)u = 0,$$

$$(2) \quad v'' + \gamma(x)v = 0.$$

If $\gamma(x) \geq c(x)$, then solutions of (2) oscillate more rapidly than solutions of (1). More precisely, if $u(x)$ is a nontrivial solution of (1) for which $u(x_1) = u(x_2) = 0$ ($x_1 < x_2$) and $\gamma(x) \geq c(x)$ for $x_1 \leq x \leq x_2$, then $v(x)$ has a zero in $(x_1, x_2]$.

This basic result has numerous generalizations, the following of which will be relevant to this paper.

1. It is possible to replace the condition $u(x_1) = 0$ by $u'(x_1) + \sigma u(x_1) = 0$ where σ is a constant ($-\infty \leq \sigma < \infty$) and $\sigma = -\infty$ is used to denote the condition $u(x_1) = 0$. In this case one concludes that every solution of (2) satisfying $v'(x_1) + \tau v(x_1) = 0$ with $\tau \geq \sigma$ has a zero in $(x_1, x_2]$.

2. The pointwise inequality $\gamma(x) \geq c(x)$ can be replaced by weaker integral inequalities. Such weaker conditions have been established by several authors, but of special interest to us here are the results of Levin [1] which deal with nontrivial solutions of (1) and (2) satisfying

$$(3) \quad u'(x_1) + \sigma u(x_1) = 0,$$

$$(4) \quad v'(x) + \tau v(x_1) = 0,$$

respectively.

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THEOREM 1 (LEVIN). *If σ and τ are finite, $u(x_2)=0$ for some $x_2 > x_1$, and if*

$$\tau + \int_{x_1}^x \gamma(t) dt \geq \left| \sigma + \int_{x_1}^x c(t) dt \right|$$

for $x_1 \leq x \leq x_2$, then $v(x)$ has a zero in $(x_1, x_2]$.

3. Sturmian theorems have been studied for the general *nonselfadjoint* linear differential equation of second order. Such results have been established by Kamke [2] using the Pruefer transformation and by Swanson [3] and the author [4] as special cases of Sturmian theorems for nonselfadjoint elliptic equations. However the hypotheses required in these theorems were either pointwise inequalities or integral inequalities of a type different from Levin's and involving the solution $u(x)$ of the differential equation which "oscillates slower."

The purpose of this paper is to unify the three generalizations indicated above for the case of nonselfadjoint equations of the form

$$(5) \quad u'' - 2b(x)u' + c(x)u = 0,$$

$$(6) \quad v'' - 2\beta(x)v' + \gamma(x)v = 0$$

whose coefficients are assumed to be real and continuous. (The general linear second order differential equation can always be put into such form by use of a Liouville transformation.) By means of the transformation

$$w = -u'/u; \quad z = -v'/v;$$

the equations (5) and (6) are transformed into Riccati equations

$$(5') \quad w' = w^2 + 2bw + c,$$

$$(6') \quad z' = z^2 + 2\beta z + \gamma$$

and the initial conditions

$$(7) \quad -u'(x_1)/u(x_1) = \sigma; \quad -v'(x_1)/v(x_1) = \tau$$

for (5) and (6) become initial values

$$(8) \quad w(x_1) = \sigma; \quad z(x_1) = \tau$$

for (5') and (6'). The differential equations (5') and (6') subject to (8) can in turn be written as integral equations

$$(5'') \quad w(x) = \sigma + \int_{x_1}^x w^2 dt + \int_{x_1}^x 2bw dt + \int_{x_1}^x c dt,$$

$$(6'') \quad z(x) = \tau + \int_{x_1}^x z^2 dt + \int_{x_1}^x 2\beta z dt + \int_{x_1}^x \gamma dt.$$

It is obvious from these equations that if $\tau \geq \sigma \geq 0$, $\beta(x) \geq b(x) \geq 0$, and

$$\int_{x_1}^x \gamma(t) dt \geq \int_{x_1}^x c(t) dt \geq 0$$

on an interval $[x_1, x_2]$, then $z(x) \geq w(x) \geq 0$ as long as $z(x)$ can be continued on $[x_1, x_2]$. Since the singularities of $w(x)$ and $z(x)$ correspond to the zeros of $u(x)$ and $v(x)$, respectively, these observations lead to the following elementary comparison theorem for (5) and (6).

THEOREM 2. *Suppose $u(x)$ is a nontrivial solution of (5) satisfying $-u'(x_1)/u(x_1) = \sigma \geq 0$, $u(x_2) = 0$. If*

$$(i) \quad \hat{\beta}(x) \geq b(x) \geq 0 \quad \text{for } x_1 \leq x \leq x_2,$$

$$(ii) \quad \int_{x_1}^x \gamma(t) dt \geq \int_{x_1}^x c(t) dt \geq 0 \quad \text{for } x_1 \leq x \leq x_2,$$

then every solution of (6) satisfying $-v'(x_1)/v(x_1) \geq \sigma$ has a zero in $(x_1, x_2]$.

We shall be interested in variations of Theorem 2 which do not require the nonnegativity of σ , τ , $b(x)$, and $\int_{x_1}^x c(t) dt$. To that end we note that the integral equations (5'') and (6'') can be written

$$(5'') \quad w(x) = \sigma + \int_{x_1}^x (w + b)^2 dt + \int_{x_1}^x (c - b^2) dt,$$

$$(6'') \quad z(x) = \tau + \int_{x_1}^x (z + \beta)^2 dt + \int_{x_1}^x (\gamma - \beta^2) dt.$$

This formulation shows that condition (ii) of Theorem 2 can be replaced by

$$\int_{x_1}^x (\gamma - \beta^2) dt \geq \int_{x_1}^{x_2} (c - b^2) dt \geq 0.$$

It also allows for other results of a more general nature.

LEMMA 1. *Let $w(x)$ and $z(x)$ be solutions of (5'') and (6''), respectively, for which $\sigma > -\infty$ and*

$$(i) \quad \tau + \int_{x_1}^x (\gamma - \beta^2) dt > \left| \sigma + \int_{x_1}^x (c - b^2) dt \right| \quad \text{for } x_1 \leq x \leq x_2,$$

$$(ii) \quad \beta(x) \geq |b(x)| \quad \text{for } x_1 \leq x \leq x_2.$$

Then $z(x) > |w(x)|$ as long as $z(x)$ can be continued on $[x_1, x_2]$.

PROOF. From (6'') we have $z(x) \geq \tau + \int_{x_1}^x (\gamma - \beta^2) dt$ for $x_1 \leq x \leq x_2$. Using

(i) and (5'') this implies that

$$\begin{aligned} z(x) &> -\sigma - \int_{x_1}^x (c - b^2) dt \\ &> -\sigma - \int_{x_1}^x (c - b^2) dt - \int_{x_1}^x (w + b)^2 dt > -w(x) \end{aligned}$$

for $x_1 \leq x \leq x_2$. It remains to show that $z(x) > w(x)$. To that end we assume to the contrary that there exists $x_0 \in (x_1, x_2]$ such that $z(x_0) \leq w(x_0)$. Then there exists an $\tilde{x} \in (x_1, x_0]$ such that $z(\tilde{x}) = w(\tilde{x})$ and $z(x) > |w(x)|$ for $x_1 \leq x < \tilde{x}$. Using (ii) we have that

$$z(x) + \beta(x) > |w(x)| + |b(x)| \geq |w(x) + b(x)| \quad \text{for } x_1 \leq x < \tilde{x},$$

and consequently that $\int_{x_1}^{\tilde{x}} (z + \beta)^2 dt > \int_{x_1}^{\tilde{x}} (w + b)^2 dt$. Using (6''), (i), and (5'') yields

$$\begin{aligned} w(\tilde{x}) &= \sigma + \int_{x_1}^{\tilde{x}} (c + b^2) dt + \int_{x_1}^{\tilde{x}} (w + b)^2 dt \\ &< \tau + \int_{x_1}^{\tilde{x}} (\gamma - \beta^2) dt + \int_{x_1}^{\tilde{x}} (z + \beta)^2 dt = z(\tilde{x}) \end{aligned}$$

which is a contradiction and establishes the lemma.

A continuity argument can now be used to establish the following.

LEMMA 2. *Let $w(x)$ and $z(x)$ be solutions of (5'') and (6''), respectively, for which $\sigma > -\infty$ and*

$$(i) \quad \tau + \int_{x_1}^x (\gamma - \beta^2) dt \geq \left| \sigma + \int_{x_1}^x (c - b^2) dt \right| \quad \text{for } x_1 \leq x \leq x_2,$$

$$(ii) \quad \beta(x) \geq |b(x)| \quad \text{for } x_1 \leq x \leq x_2.$$

Then $z(x) \geq w(x)$ as long as $z(x)$ can be continued on $[x_1, x_2]$.

As an immediate consequence of Lemma 2 we have the following generalization of Levin's Theorem 1.

THEOREM 3. *Suppose $u(x)$ and $v(x)$ are nontrivial solutions of (5) and (6), respectively, and that $u(x) \neq 0$ for $x_1 \leq x < x_2$, $u(x_2) = 0$. If*

$$(i) \quad -\frac{v'(x_1)}{v(x_1)} + \int_{x_1}^x (\gamma - \beta^2) dt \geq \left| -\frac{u'(x_1)}{u(x_1)} + \int_{x_1}^x (c - b^2) dt \right|$$

for $x_1 \leq x \leq x_2$,

$$(ii) \quad \beta(x) \geq |b(x)| \quad \text{for } x_1 \leq x \leq x_2,$$

then $v(x)$ has a zero in $(x_1, x_2]$.

Theorems 2 and 3 show how to estimate zeros of the nonselfadjoint equations (6) in terms of the coefficients and the initial value of $-v'(x_1)/v(x_1)$. Furthermore Theorem 2 shows that such estimates are sometimes simpler in case $\tau \equiv -v'(x_1)/v(x_1)$ is nonnegative. The following result shows how one can "shift" the initial value of $-v'(x_1)/v(x_1)$ by means of a compensating shift in the coefficient of the differential equation.

THEOREM 4. *Let $v(x)$ be a solution of (6) satisfying $-v'(x_1)/v(x_1) = \tau$. The first zero of $v(x)$ is the same as the first zero of $V(x)$, where $V(x)$ is a solution of*

$$(9) \quad \begin{aligned} V'' - 2(\beta - \tau_0)V' + (c - 2\beta\tau_0 + \tau_0^2)V &= 0, \\ -V'(x_1)/V(x_1) &= \tau + \tau_0. \end{aligned}$$

PROOF. The substitution $z(x) = -v'(x)/v(x)$ leads to

$$z(x) = \tau + \int_{x_1}^x z^2 dt + \int_{x_1}^x 2\beta z dt + \int_{x_1}^x c dt.$$

Defining $Z(x) = z(x) + \tau_0$ yields

$$\begin{aligned} Z(x) &= \tau + \tau_0 + \int_{x_1}^x (Z - \tau_0)^2 dt + \int_{x_1}^x 2\beta(Z - \tau_0) dt + \int_{x_1}^x c dt \\ &= \tau + \tau_0 + \int_{x_1}^x Z^2 dt + \int_{x_1}^x 2(\beta - \tau_0)Z dt + \int_{x_1}^x (c - 2\beta\tau_0 + \tau_0^2) dt. \end{aligned}$$

Now the first singularity of $Z(x)$ coincides with the first singularity of $z(x)$ and therefore with the first zero of $v(x)$. But the first singularity of $Z(x)$ also coincides with the first zero of $V(x)$ satisfying (9), by the substitution $Z(x) = -V'(x)/V(x)$. This completes the proof.

The shift formula of Theorem 4 can also be applied in connection with other known comparison theorems for nonselfadjoint differential equations [4].

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