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ON A QUESTION OF ERDÖS CONCERNING COHESIVE BASIC SEQUENCES

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ABSTRACT. For an arbitrary basic sequence $\mathscr{B}$, set $V(\mathscr{B})=\{{}^{\#}B_k \mid k\in Z^+\}$, where ${}^{\#}B_k$ is the number of pairs (a, b) in $\mathscr{B}$ such that $ab=k$. It is proved that $V(\mathscr{B})$ is unbounded if either $\mathscr{B}$ is cohesive or $\mathscr{B}\not\subset\mathscr{M}$. The set $V(\mathscr{B})$ is determined explicitly in these cases.

A *basic sequence* $\mathscr{B}$ is a set of pairs (a, b) of positive integers satisfying

(i) if $(a, b)\in\mathscr{B}$, then $(b, a)\in\mathscr{B}$;

(ii) $(a, bc)\in\mathscr{B}$ if and only if $(a, b)\in\mathscr{B}$ and $(a, c)\in\mathscr{B}$;

(iii) $(1, k)\in\mathscr{B}$ for every positive integer k.

We denote by B_k the set of pairs (a, b) in $\mathscr{B}$ for which $ab=k$, and we denote by $\mathscr{M}$ the familiar basic sequence consisting of all pairs (a, b) of relatively prime positive integers.

The *density* of a basic sequence $\mathscr{B}$ is defined to be

$$\delta(\mathscr{B}) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \frac{{}^{\#}B_k}{d(k)},$$

if the limit exists, where ${}^{\#}B_k$ is the number of pairs in B_k, and $d(k)$ is the number of positive divisors of k.

A basic sequence $\mathscr{B}$ is said to be *cohesive* if, for each positive integer k, there is an integer $a=a(k)>1$ such that $(a, k)\in\mathscr{B}$.

If $\mathscr{B}$ is cohesive and $\mathscr{B}\subset\mathscr{M}$, then for each prime p there are infinitely many primes q such that $(p, q)\in\mathscr{B}$ (see [**2**, Corollary 4.2.1]). One might expect, therefore, that for such basic sequences the values of ${}^{\#}B_k$ ($k=1, 2, \cdots$) would be fairly large. It has been shown, however, that there do exist cohesive basic sequences $\mathscr{B}$, with $\mathscr{B}\subset\mathscr{M}$, for which $\delta(\mathscr{B})=0$ (see [**3**]). Following the sense of this last result, Paul Erdös asked recently whether there are any cohesive basic sequences $\mathscr{B}$ contained in $\mathscr{M}$ for which the set

$$V(\mathscr{B}) = \{{}^{\#}B_k \mid k = 1, 2, \cdots\}$$

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is bounded. (These basic sequences would obviously have the property that $\delta(\mathscr{B})=0$.)

We will show here that no such basic sequences exist; in fact, we will determine exactly the set $V(\mathscr{B})$ for any cohesive basic sequence.

THEOREM. (a) *If $\mathscr{B}$ is any basic sequence (cohesive or not) which is not contained in $\mathscr{M}$, then $V(\mathscr{B})=Z^+$ (the set of positive integers).*

(b) *If $\mathscr{B}\subset\mathscr{M}$, then $V(\mathscr{B})\subset\{2^m|m=0,1,2,\cdots\}$.*

(c) *If $\mathscr{B}\subset\mathscr{M}$ and $\mathscr{B}$ is cohesive, then*

$$V(\mathscr{B}) = \{2^m \mid m = 0, 1, 2, \cdots\}.$$

PROOF. If $\mathscr{B}\not\subset\mathscr{M}$, then $(p,p)\in\mathscr{B}$ for some prime p. For $k=p^\alpha$ $(\alpha=0,1,2,\cdots)$,

$$B_k = \{(1, p^\alpha), (p, p^{\alpha-1}), \cdots, (p^\alpha, 1)\},$$

so ${}^\#B_k=\alpha+1$. Therefore $V(\mathscr{B})=Z^+$. That proves (a).

Suppose now that $\mathscr{B}\subset\mathscr{M}$. Let $n=p_1^{\alpha_1}\cdots p_N^{\alpha_N}$. The proof that ${}^\#B_n=2^m$ for some m in Z^+ depends on a very simple modification of the concept of severance classes developed by Carroll and Gioia **[1]**.

We say that two distinct prime divisors p and q of n are *severed in n with respect to $\mathscr{B}$* if there is a chain

$$p = q_1, q_2, q_3, \cdots, q_l = q$$

of prime divisors of n such that

$$(q_i, q_{i+1}) \notin \mathscr{B}, \qquad (i = 1, 2, \cdots, l-1).$$

For each prime divisor p of n, let $S(p)$ consist of the prime p together with all the prime divisors q of n which are severed from p in n with respect to $\mathscr{B}$.

The sets $S(p_1),\cdots,S(p_N)$ clearly form a partition of $\{p_1,\cdots,p_N\}$. We may suppose that there are M distinct partition sets; say

$$H = \{S(p_1), S(p_2), \cdots, S(p_M)\}$$

is the collection of these sets.

From the construction of $S(p_1),\cdots,S(p_M)$, one may prove that

(1) if $p\in S(p_i)$ and $q\in S(p_j)$ with $i\neq j$, then $(p,q)\in\mathscr{B}$;

(2) if $(a,b)\in B_n$, then all the primes in a given partition set $S(p_i)$ will divide a, or all will divide b. No pair of prime divisors p of a and q of b can occur in the same partition set.

Statement (1) follows from the fact that if (p,q) were not in $\mathscr{B}$, then p_i and p_j would be severed in n with respect to $\mathscr{B}$, and so $S(p_i)$ and $S(p_j)$ would be the same.

To prove (2), let us suppose that $(a, b) \in \mathscr{B}$, $ab=n$, $p|a$, $q|b$, and that p and q are in the same partition set. We will arrive at a contradiction.

Since p and q are severed in n with respect to $\mathscr{B}$, there are primes $p=r_1, r_2, \cdots, r_l=q$, each a divisor of n and hence also a divisor of a or b, such that $(r_i, r_{i+1}) \notin \mathscr{B}$ $(i=1, 2, \cdots, l-1)$. Let i_0 be the largest index such that $r_{i_0}|a$ ($r_1=p$, so $r_1|a$ and i_0 exists).

Now we assert that $i_0<l$. For if $i_0=l$, then $r_{i_0}=r_l=q$, and then q would divide both a and b. But this is not possible since $(a, b) \in \mathscr{B} \subset \mathscr{M}$, so a and b are relatively prime.

Thus $i_0+1 \leqq l$ and so $r_{i_0+1}|b$. But now we have $r_{i_0}|a$, $r_{i_0+1}|b$, $(a, b) \in \mathscr{B}$, which means $(r_{i_0}, r_{i_0+1}) \in \mathscr{B}$, and this contradiction proves statement (2).

It follows from (1) and (2) that ${}^{\#}B_n$ is equal to the number of subsets of H, for $(a, b) \in B_n$ if and only if a has as prime divisors all those primes contained in some (possibly empty) subcollection of the partition sets in H, and b has as prime divisors all those primes contained in the remaining partition sets in H. Hence ${}^{\#}B_n=2^M$.

Since ${}^{\#}B_1=1$, assertion (b) is proved.

Finally, suppose $\mathscr{B}$ is cohesive and $\mathscr{B} \subset \mathscr{M}$. ${}^{\#}B_1=1$, so let m be an arbitrary positive integer. Let p_1 be an arbitrary prime. Since $\mathscr{B}$ is cohesive, there is a prime p_2 such that $(p_1, p_2) \subset \mathscr{B}$. Since $\mathscr{B} \subset \mathscr{M}$, $p_1 \neq p_2$. Again since $\mathscr{B}$ is cohesive, there is a prime p_3 such that $(p_1p_2, p_3) \in \mathscr{B}$, and since $\mathscr{B} \subset \mathscr{M}$, we have $p_3 \neq p_1, p_2$. In this way we may select m distinct primes $p_1, \cdots, p_m$ such that $(p_i, p_j) \in \mathscr{B}$ for $i \neq j$. If $k=p_1 \cdots p_m$, then ${}^{\#}B_k=2^m$. That proves (c) and completes the proof of the theorem.

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