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## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# MEAN CONVERGENCE IN $L^p$ SPACES

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ABSTRACT. Let $(X, \mathscr{B}, \mu)$ be a measure space and $\{f_n\}$ be a sequence in $L^p(X, \mathscr{B}, \mu)$, $0<p<\infty$. The author presents a very short proof of the familiar fact that if $f_n \to f$ $\mu$-a.e. and $\|f_n\|_p \to \|f\|_p < \infty$, then $\|f_n - f\|_p \to 0$.

In this note we offer an elementary and especially brief proof of a standard result from the abstract Lebesgue theory. The following theorem appears, for example, in [**1**, Lemma 1, p. 21], in [**2**, pp. 208–209], and in [**3**, Theorem 13, p. 34].

THEOREM. *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\{f_n\}$ be a sequence in $L^p(X, \mathscr{B}, \mu)$, $0<p<\infty$. If $f_n \to f$ $\mu$-a.e. and $\|f_n\|_p \to \|f\|_p < \infty$, then $\|f_n - f\|_p \to 0$.*

PROOF. It is an elementary fact that if $a$, $b$ are nonnegative real numbers and $0<p<\infty$, then $(a+b)^p \leqq 2^p(a^p+b^p)$. It follows that $2^p(|f_n|^p+|f|^p)-|f_n-f|^p$, $n=1, 2, \cdots$, is a sequence of nonnegative measurable functions which clearly converges pointwise $\mu$-a.e. to the integrable function $2^{p+1}|f|^p$. By Fatou's lemma,

$$2^{p+1}\int |f|^p \, d\mu \leqq \liminf \int [2^p(|f_n|^p + |f|^p) - |f_n - f|^p] \, d\mu$$
$$= 2^{p+1}\int |f|^p \, d\mu - \limsup \int |f_n - f|^p \, d\mu.$$

This implies that $\limsup \|f_n - f\|_p^p \leqq 0$ and we are finished.

REMARKS. All other proofs known to us are not only more lengthy but require, in addition, Egorff's theorem. For an important application of the theorem just proved see [**1**, Theorem 2.6, p. 21] where F. Riesz's theorem on mean convergence of $H^p$ functions to their boundary function is established.

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