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THE MORSE LEMMA ON BANACH SPACES

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ABSTRACT. Let $f: U \to R$ be a C^3 map of an open subset U of a Banach space E. Let $p \in U$ be a critical point of f ($df_p=0$). If E is a conjugate space ($E=F^*$) we define what it means for p to be nondegenerate. In this case there is a diffeomorphism γ of a neighborhood of p with a neighborhood of $0 \in E$, $\gamma(p)=0$ with

$$f \circ \gamma^{-1}(x) = \tfrac{1}{2}d^2f_p(x, x) + f(p).$$

In this paper we define a notion of nondegeneracy for critical points of smooth real valued functions defined on open subsets of a Banach space E which is a dual space, and we shall prove the Morse lemma with our definition. Earlier notions of nondegeneracy which implied the Morse lemma are stronger than ours and in particular had the unfortunate consequence of implying that $E \approx E^*$. See Palais **[3]** and **[4]** for the case E=Hilbert space and $E \approx E^*$ respectively.

DEFINITION. Let E_0 be a Banach space and let $E=E_0^*$ be the dual space of E_0. Denote by $\langle\ ,\ \rangle: E \times E_0$ the bilinear pairing of E_0 and E given by $\langle f, x\rangle = f(x)$.

In the paragraphs below we shall not be too careful about the order of the terms in $\langle\ ,\ \rangle$; e.g., $\langle x, f\rangle$ instead of $\langle f, x\rangle$ and no confusion should arise from this.

A continuous linear map $A \in L(E, E_0)$ is said to be *symmetric* if $\langle y, Ax\rangle = \langle x, Ay\rangle$ for all $x, y \in E$. We note this by writing $A \in L_s(E, E_0)$. The proof of the following is standard.

LEMMA 1. *If $A \in L_s(E, E_0)$ is injective then $A(E)$ is dense in E_0.*

DEFINITION. Let $U \subset E$ be an open subset with $f: U \to R$ C^2-differentiable. We say that f is weak (*) smooth if for each fixed $t \in U$ and $h \in E$ the linear functional on E given by $k \to d^2f_t(h, k)$ is continuous in the weak (*) topology of E induced by E_0. If E is reflexive every C^2-map on U is weak (*) smooth.

LEMMA 2. *Let $t \in U$ be fixed with $f: U \to R$ C^2 and weak (*) smooth. Then d^2f_t induces a map $A_t \in L_s(E, E_0)$ with $k(A_t(h)) = \langle k, A_t(h)\rangle = d^2f_t(h, k)$.*

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PROOF. Since $k \to d^2f_t(h, k)$ is continuous in the weak (*) topology it is given by an element of E_0, say $A_t(h)$ according to the rule $k(A_t(h)) = d^2f_t(h, k)$. (For a proof of this fact see [1, p. 421].) A_t is readily seen to be linear in h since $d^2f_t(h, k)$ is. This completes the proof.

Let U be a convex neighborhood of a critical point p of f: $U \to R$ (i.e., $df_p = 0$) which we assume to be C^3 and weak (*) smooth. Then by Taylor's formula for $x \in U$, $x = p + \xi$,

$$f(p + \xi) = f(p) + \int_0^1 (1 - \lambda) d^2f(p + \lambda\xi)(\xi, \xi)\, d\lambda.$$

Let $B_x : E \times E \to R$ be the symmetric bilinear form given by

$$B_x(h, k) = \int_0^1 (1 - \lambda) d^2f(p + \lambda\xi)(h, k)\, d\lambda.$$

Then, by the above remarks, $B_x(h, k) = \langle k, A_x h\rangle$ where $A_x \in L_s(E, E_0)$. $A : U \to L_s(E, E_0)$ is called the bilinear representative of f in U. If f is C^3 then A is C^1. If $p = 0$ the above Taylor expansion for f can then be written

$$f(x) = B_x(x, x) + f(0) = \langle A_x(x), x\rangle + f(0)$$

for x near 0. In the following paragraphs we shall assume for simplicity that our critical point p is in fact 0.

DEFINITION. The critical point p is *nondegenerate* if there is a neighborhood $W \subset U$ and positive constants C_1 and C_2 with

(1) A_0 injective,

(2) $\|DA_t(h)(y)\| \leqq C_1 \|h\|\ \|A_{t'}y\|$,

(3) $\|DA_{t_1}(h)(y) - DA_{t_2}(h)(y)\| \leqq C_2 \|h\| \cdot \|t_1 - t_2\| \cdot \|A_{t'}y\|$,

for all $t, t', t_1, t_2 \in W$, where DA_t denotes the Fréchet derivative of A with respect to the subscript variable.

REMARK. In the case $E = H$, Hilbert space, considered by Palais and Smale, $A_0 \in L_s(H, H)$ was required to be an isomorphism for p to be a nondegenerate critical point of f. The requirement that A_0 be an isomorphism implies (1), (2), and (3), and so these conditions are weaker than the nondegeneracy condition of Palais and Smale.

LEMMA 3. *Let A be a bilinear representative of f at p. Then there exists a neighborhood W of 0 and constant $K > 1$ with the property that*

$$K^{-1} \|A_0 y\| \leqq \|A_t y\| \leqq K \|A_0 y\|.$$

PROOF. By the mean value theorem we have

$$A_t y - A_0 y = \int_0^1 DA_{\lambda t}(t)(y)\, d\lambda.$$

Let W be a bounded convex neighborhood of $0 \in E$ where the nondegeneracy conditions hold. Then

$$\|A_t - A_0 y\| \leqq \int_0^1 \|DA_{\lambda t}(t)(y)\| \, d\lambda \leqq C_1 \|t\| \cdot \|A_{t'} y\|$$

for all $t, t' \in W$.

From this (dividing by $\|A_{t'}y\|$), we can conclude that

$$\left| \frac{\|A_t y\|}{\|A_{t'} y\|} - \frac{\|A_0 y\|}{\|A_{t'} y\|} \right| \leqq C_1 \|t\|.$$

Then setting $t'=0$ we get $\|A_t y\| \leqq (1+C_1\|t\|)\|A_0 y\|$ and setting $t'=t$ we get $\|A_0 y\| \leqq (1+C_1\|t\|)\|A_t y\|$. Finally, setting $K=1+\sup_{t\in W} C_1\|t\|$ concludes the proof.

REMARK. The above lemma implies that the map $A: W \to L_s(E, E_0)$ takes values A_t which are injective maps for all $t \in W$.

LEMMA 4. *Let A be a bilinear representative of f at p. Then there exists a neighborhood W of 0 and constants C_1 and C_2 with*

$$\|A_{t_1} y - A_{t_2} y\| \leqq C_1 \|t_1 - t_2\| \cdot \|A_{t'} y\|$$

for $t_1, t_2, t' \in W$.

PROOF. Follows directly from the mean value theorem.

LEMMA 5. *Let $f: \mathcal{O} \to R$ be a real valued C^3 weak (*) smooth function defined in the neighborhood $\mathcal{O}$ of the origin of E. Suppose 0 is a nondegenerate critical point of f. From before*

$$f(x) = \langle A_x(x), x \rangle + f(0).$$

Then there exists a C^1 map $t \to Q_t \in GL(E)$, the general linear group of E with $A_t Q_t = A_0$ and $Q_0 = I$.

PROOF. Let W be a convex neighborhood of 0 so that Lemmas 3 and 4 hold. Let $t \in W$ be fixed. Then by the remark after Lemma 3, A_t is injective; moreover, Range A_t is dense in E_0. Let $y \in E$. Define $Q_t(y) \in E_0^*$ by

$$\langle Q_t(y), A_t(x) \rangle = Q_t(y)(A_t x) = \langle y, A_0 x \rangle.$$

By Lemma 3, $Q_t(y)$ is continuous. Since Range A_t is dense this defines $Q_t(y)$ on all of E_0. It is easy to check that for each $t \in W$, that Q_t is linear. We shall show that Q_t is an isomorphism and that the map $t \to Q_t$ is C^1.

First, since A_t is symmetric for each $t \in W$, $\langle A_t Q_t y, x \rangle = \langle A_0 y, x \rangle$. Therefore,

$$|\langle A_t x, Q_t y \rangle| = |\langle A_t Q_t y, x \rangle| = |\langle A_0 x, y \rangle| \leqq K \|y\| \cdot \|A_t x\|.$$

Thus,

$$|\langle Q_t y, A_t x/\|A_t x\|\rangle| \leqq K\,\|y\|$$

for all $x \in E$. This implies that $\|Q_t y\| \leqq K\|y\|$. (Recall for $f \in E_0^*$, $\|f\| = \sup_{\|x\|=1} |\langle f, x\rangle|$ and if $S \subset E_0$ is dense $\|f\| = \sup_{\|x\|=1; x\in S} |\langle f, x\rangle|$.) Secondly,

$$|\langle A_0 x, y\rangle| = |\langle A_0 y, x\rangle| = |\langle A_t Q_t y, x\rangle| \leqq \|Q_t y\| \cdot \|A_t x\| \leqq K\,\|Q_t y\| \cdot \|A_0 x\|.$$

Thus

$$|\langle A_0 x/\|A_0 x\|, y\rangle| \leqq K\,\|Q_t y\|$$

for all $x, y \in E$, which implies that $\|Q_t y\| \geqq K^{-1}\|y\|$, which further implies that for each $t \in W$, Q_t is injective, continuous and has closed range. To see that Q_t is invertible, let $y \in E$ be arbitrary. Define $P_t(y) \in E_0^*$ by $P_t(y)(A_0 x) = \langle A_t x, y\rangle$. Then

$$\langle A_0 P_t y, x\rangle = \langle A_t y, x\rangle.$$

As before, P^t will be continuous, injective and have closed range. Now

$$\langle A_t Q_t P_t y, x\rangle = \langle A_0 P_t y, x\rangle = \langle A_t y, x\rangle,$$

for all x, y. This implies that $A_t Q_t P_t = A_t$ and since A_t is injective, we have that $Q_t P_t = I$, which shows that Q_t is onto, and hence invertible for each t.

We must now show that the map $t \to Q_t$ is C^1. The continuity of this map is guaranteed by Lemma 4. For

$$\begin{aligned} |\langle Q_{t_1} y - Q_{t_2} y, A_{t_1} x\rangle| &= |\langle Q_{t_1} y, A_{t_1} x\rangle - \langle Q_{t_2} y, A_{t_1} x\rangle| \\ &= |\langle A_0 y, x\rangle - \langle Q_{t_2} y, A_{t_1} y\rangle| \\ &= |\langle Q_{t_2} y, A_{t_2} x\rangle - \langle Q_{t_2} y, A_{t_1} x\rangle| \\ &= |\langle Q_{t_2} y, A_{t_2} x - A_{t_1} x\rangle| \leqq \|Q_{t_2} y\| \cdot \|A_{t_2} x - A_{t_1} x\| \\ &\leqq KC_1\,\|y\| \cdot \|t_1 - t_2\| \cdot \|A_{t_1} x\|. \end{aligned}$$

Thus

$$\|Q_{t_1} y - Q_{t_2} y\| \leqq C_1 K\,\|y\| \cdot \|t_1 - t_2\|$$

or

$$\|Q_{t_1} - Q_{t_2}\| \leqq C_1 K\,\|t_1 - t_2\|$$

which gives the continuity of the map $t \to Q_t$.

To see that the map is smooth, we use the same sort of trick we have already employed. Our candidate for the derivative of Q_t will be the map defined by

$$\langle DQ_t(h)(y), A_t x\rangle = -\langle DA_t(h)(x), Q_t y\rangle.$$

The nondegeneracy assumption assures us that the map DQ_t is well defined and continuous.

Let $t_0 \in W$ be fixed and let h be so small that $t_0+h \in W$. Then

$$(*) \qquad \begin{aligned}\langle Q_{t_0+h}y - Q_{t_0}y, A_{t_0}x\rangle &= \langle Q_{t_0+h}y, A_{t_0}x\rangle - \langle Q_{t_0}y, A_{t_0}x\rangle \\ &= \langle Q_{t_0+h}y, A_{t_0}x - A_{t_0+h}x\rangle.\end{aligned}$$

By the Fréchet differentiability of $t \to A_t$ this is equal to

$$\langle Q_{t_0+h}y, -DA_{t_0}(h)(x) - R_{t_0}(h)(x)\rangle$$

where

$$A_{t_0+h} - A_{t_0} = DA_{t_0}(h) + R_{t_0}(h)$$

and

$$R_{t_0}(h)(x) = \int_0^1 [DA_{t_0+\lambda h}(h)(x) - DA_{t_0}(h)(x)]\, d\lambda.$$

(This implies that $\|R_{t_0}(h)(x)\| \leqq C_2\|h\|^2 \cdot \|A_{t_0}x\|$.) Continuing the string of equalities we get (*) equal to

$$\begin{aligned}&\langle Q_{t_0+h}(y), -DA_{t_0}(h)(x)\rangle - \langle Q_{t_0+h}(y), R_{t_0}(h)(x)\rangle \\ &\qquad = -\langle Q_{t_0}(y), DA_{t_0}(h)(x)\rangle + \langle Q_{t_0+h}(y) - Q_{t_0}(y), -DA_{t_0}(h)(x)\rangle \\ &\qquad\quad -\langle Q_{t_0+h}(y), R_{t_0}(h)(x)\rangle \\ &\qquad = \langle DQ_{t_0}(h)(y), A_{t_0}x\rangle + \langle Q_{t_0+h}(y) - Q_{t_0}(y), -DA_{t_0}(h)(x)\rangle \\ &\qquad\quad -\langle Q_{t_0+h}(y), R_{t_0}(h)(x)\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&|\langle Q_{t_0+h}y - Q_{t_0}y - DQ_{t_0}(h)(y), A_{t_0}(x)\rangle| \\ &\qquad \leqq |\langle Q_{t_0+h}(y) - Q_{t_0}(y), DA_{t_0}(h)(x)\rangle| + |\langle Q_{t_0+h}(y), R_{t_0}(h)(x)\rangle|.\end{aligned}$$

By the nondegeneracy conditions, we get that these two terms are bounded by

$$\begin{aligned}&\leqq \{C_1K\,\|h\|\cdot\|y\|\}\cdot\{\|DA_{t_0}(h)(x)\|\} + \{K\,\|y\|\}\cdot\{C_2\,\|h\|^2\,\|A_{t_0}x\|\} \\ &\leqq \{C_1KC_2 + KC_2\}\,\|h\|^2\,\|y\|\cdot\|A_{t_0}x\|.\end{aligned}$$

Thus

$$\|Q_{t_0+h}y - Q_{t_0}y - DQ_{t_0}(h)(y)\| \leqq \{C_1C_2K + KC_2\}\,\|h\|^2\,\|y\|$$

which implies that

$$\|Q_{t_0+h} - Q_{t_0} - DQ_{t_0}(h)\| \leqq \{C_1C_2K + KC_2\}\,\|h\|^2.$$

Therefore

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|}\{\|Q_{t_0+h} - Q_{t_0} - DQ_{t_0}(h)\|\} = 0$$

and hence $t \to Q_t$ is Fréchet differentiable.

The last thing to show is that $t \to DQ_t$ is continuous. Now

$$\begin{aligned}
&\langle DQ_{t_1}(h)(y) - DQ_{t_2}(h)(y), A_{t_1}x\rangle \\
&\quad = -\langle Q_{t_1}y, DA_{t_1}(h)(x)\rangle - \langle DQ_{t_2}(h)(y), A_{t_1}x\rangle \\
&\quad = -\langle Q_{t_1}y, DA_{t_1}(h)(x)\rangle - \langle DQ_{t_2}(h)(y), A_{t_2}x\rangle \\
&\qquad + \langle DQ_{t_2}(h)(y), A_{t_2}x - A_{t_1}x\rangle \\
&\quad = -\langle Q_{t_1}y, DA_{t_1}(h)(x) - DA_{t_2}(h)(x)\rangle + \langle DQ_{t_2}(h)(y), A_{t_2}x - A_{t_1}x\rangle.
\end{aligned}$$

As before

$$\begin{aligned}
|\langle DQ_{t_1}(h)(y) - DQ_{t_2}(h)(y), A_{t_1}(x)\rangle| &\leqq \{K\,\|y\|\}\{C_2\,\|h\|\cdot\|t_1 - t_2\|\cdot\|A_{t_1}x\|\} \\
&\quad + \{\|DQ_{t_2}(h)(y)\|\cdot[C_1]\cdot\|t_1 - t_2\|\cdot\|A_{t_1}x\|\} \\
&\leqq \{KC_2 + KC_2C_1\}\{\|h\|\cdot\|y\|\cdot\|t_1 - t_2\|\cdot\|A_{t_1}x\|\}
\end{aligned}$$

which implies that

$$\|DQ_{t_1} - DQ_{t_2}\| \leqq \{KC_2 + KC_2C_1\}\,\|t_1 - t_2\|$$

which concludes the proof of Lemma 5.

THEOREM (MORSE LEMMA). *Let $f:\mathcal{O}\to R$ be a C^3 and weak (*) smooth map with $0\in\mathcal{O}$ a nondegenerate critical point of f. Then there exists a local C^1 diffeomorphism ψ preserving the origin with*

$$f\circ\psi(x) = \tfrac{1}{2}d^2f_0(x, x) + f(0) = \langle A_0x, x\rangle + f(0).$$

PROOF (FOLLOWING PALAIS [3]). By Lemma 5 here is a C^1 map $t\mapsto Q_t\in \mathrm{GL}(E)$ with $A_tQ_t = A_0$. Thus $Q_t^*A_t^* = A_0^*$ where $Q_t^*\in\mathrm{GL}(E^*)$. Since A_0 is symmetric, $(A_0^*x)y = x(A_0y) = y(A_0x) = (iA_0x)y$ for all $x, y\in E$, where $i:E_0\to E^*$ is the natural inclusion. Thus $A_0^* = iA_0$. Therefore

$$(\ast\!\ast)\qquad Q_t^*A_t^* = iA_tQ_t.$$

Also, $Q_0 = I$ and thus, by the Taylor expansion for the squre root, Q_t has a C^1 square root S_t in some neighborhood V_0 of the origin. Now since equation $(\ast\!\ast)$ is satisfied by S_t (in fact by a polynomial in Q_t or a limit of such), we have $S_t^*A_t^* = iA_tS_t$. Hence $S_t^*A_t^*S_t = iA_tS_t^2 = iA_0$ and consequently

$$A_t^* = R_t^*(iA_0)R_t$$

where $R_t = S_t^{-1}$. From the bilinear representation of f we have

$$\begin{aligned}
f(x) - f(0) = \langle A_x(x), x\rangle = x(A_x(x)) &= (A_x^*(x))(x) = (R_x^*iA_0R_x(x))(x) \\
&= (iA_0R_x(x))(R_xx) = R_x(x)(A_0R_x(x)) = \langle R_x(x), A_0R_x(x)\rangle.
\end{aligned}$$

Let $\varphi(x) = R_x(x)$. Then $D\varphi_0(h) = R_0(h) = h$, since $Q_0 = S_0 = R_0 = I$. Thus by the inverse function theorem, φ has a local inverse ψ restricted to a subneighborhood $V \subset V_0$. The map ψ clearly satisfies the requirements of the lemma.

REMARK. If we require the map ψ above to be only a local homeomorphism we can relax the nondegeneracy conditions to the following. Let $f(x)=\langle x, A_x(x)\rangle+f(p)$, $A: U \to L_s(E, E_0)$ the bilinear representative of f. Then p is nondegenerate if

(1′) A_0 is injective,

and there exists a subneighborhood $W \subset U$ of p and constant C so that

(2′) $\|A_{t_1}y - A_{t_2}y\| \leqq C\|t_1 - t_2\|\|A_{t'}y\|$

for any $t_1, t_2, t' \in W$.

We can define the notion of nondegenerate critical point of a C^3-map $f: M \to R$ on a C^3 Banach manifold M modelled on $E=E_0^*$. We say that a critical point p is nondegenerate if there is a chart (φ, U) about p, $\varphi(U)=\mathcal{O} \subset E$ with the property that $f \circ \varphi^{-1}: \mathcal{O} \to R$ is weak (*) smooth and has $\varphi(p)$ as a nondegenerate critical point.

From the last theorem we have

THEOREM. *Let $f: M \to R$ be C^3 with M modelled on $E=E_0^*$ and p a nondegenerate critical point. Then there exists a local diffeomorphism γ of a neighborhood of $\varphi(p)$ with a neighborhood of $0 \in E$, $\gamma(\varphi(p))=0$ and with*

$$f \circ (\varphi^{-1} \circ \gamma)(x) = d^2(f \circ \varphi^{-1})_{\varphi(p)}(x, x) + f(p).$$

COROLLARY. *Nondegenerate critical points are isolated.*

REMARK. It does not seem that this definition of nondegeneracy is independent of the choice of coordinate chart and hence does not appear to be a natural geometric notion of nondegeneracy for spaces E which are not isomorphic to E^*. The author is at the moment unaware of a modification of nondegeneracy for general Banach manifolds, i.e., one which is independent of the selection of coordinate chart.

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