GEÖCZE AREA AND A CONVERGENCE PROPERTY

RONALD GARIEPY

Abstract. Suppose \( f \) is a continuous mapping with finite Lebesgue area from a polyhedral region \( X \subset \mathbb{R}^k \) into \( \mathbb{R}^n, 2 \leq k \leq n \). Let \( f = l \circ m \) be the monotone-light factorization of \( f \) with middle space \( M \).

If \( f \) satisfies a "cylindrical condition" considered by T. Nishiura, then a current valued measure \( T \) over \( M \) can be associated with \( f \) by means of the Cesari-Weierstrass integral, and if \( \{ f_i \} \) is any sequence of quasi-linear maps \( f_i : X \to \mathbb{R}^n \) converging uniformly to \( f \) with bounded areas, then

\[
T(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ m)f_i^\# \phi
\]

whenever \( \phi \) is an infinitely differentiable \( k \)-form in \( \mathbb{R}^n \) and \( g \) is a continuous real valued function on \( M \) which vanishes on \( m(\text{Bdry } X) \).

The total variation measure of \( T \), taken with respect to mass, coincides with the Geöcze area measure over \( M \).

1. Suppose \( f \) is a continuous mapping with finite Lebesgue area from a polyhedral region \( X \subset \mathbb{R}^k \) into \( \mathbb{R}^n, 2 \leq k \leq n \). Let \( f = l \circ m \) denote the monotone-light factorization of \( f \) with middle space \( M \).

Suppose \( p : \mathbb{R}^n \to \mathbb{R}^k \) is an orthogonal projection. Let \( p \circ f = l \circ \tilde{m} \) denote the monotone-light factorization of \( p \circ f \) with middle space \( \tilde{M} \) and let \( h \) be the monotone map such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{m} & M \\
& \searrow & \downarrow l \\
\tilde{M} & \xrightarrow{h} & \mathbb{R}^k \\
\end{array}
\]

commutes.

Let \( C_p = M \cap \{ z : \text{diam } h^{-1}(h(z)) > 0 \} \), where diameter is taken with respect to the usual metric in \( M \). It was shown in [N1, 2.1] that, if either \( k = 2 \) or the \( k+1 \) dimensional Hausdorff measure of \( f(X) \) is zero,

Received by the editors April 16, 1971.

AMS 1970 subject classifications. Primary 26A63, 28A75.

© American Mathematical Society 1972
then the $k$ dimensional Lebesgue measure of $p \circ l(C_p)$ is zero for every orthogonal projection $p$.

Let $\Lambda(k, n)$ denote the set of all $k$-tuples $\lambda=(\lambda_1, \cdots, \lambda_k)$ of integers such that $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$ and let $p^{\lambda}: \mathbb{R}^n \to \mathbb{R}^k$ be the orthogonal projection defined by

$$p^{\lambda}(y) = (y_{\lambda_1}, \cdots, y_{\lambda_k}) \text{ for } y = (y_1, \cdots, y_n) \in \mathbb{R}^n.$$ 

Let $C(M)$ denote the space of all continuous real valued functions on $M$ and let $C_0(M)$ denote the space of those $g \in C(M)$ which vanish on $m(\text{Bdry } X)$.

Let $e_1, \cdots, e_n$ be the standard basis in $\mathbb{R}^n$ and let $e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k}$, $\lambda \in \Lambda(k, n)$, be the corresponding basis for the space of $k$-vectors in $\mathbb{R}^n$.

For $\lambda \in \Lambda(k, n)$ let $f^{\lambda} = p^{\lambda} \circ f$ and for each simple polyhedral region $\pi \subset X$ let

$$u(f^{\lambda}, \pi) = \int_{\mathbb{R}^k} O(f^{\lambda}, \pi, y) \, dy$$

where $O(f^{\lambda}, \pi, y)$ denotes the topological index of $y \in \mathbb{R}^k$ with respect to the mapping $f^{\lambda}|_{\pi}$ if $y \in \mathbb{R}^k \setminus f^{\lambda}(\text{Bdry } \pi)$ and $O(f^{\lambda}, \pi, y)=0$ if $y \in f^{\lambda}(\text{Bdry } \pi)$. Let

$$u(f, \pi) = \sum_{\lambda \in \Lambda(k, n)} u(f^{\lambda}, \pi) e_\lambda.$$ 

For any finite nonoverlapping collection $P$ of simple polyhedral regions $\pi \subset X$ let

$$\delta(P) = \max\{\text{diam } f(\pi): \pi \in P\}$$

$$+ \max\left\{V(f^{\lambda}) - \sum_{\pi \in P} |u(f^{\lambda}, \pi)|: \lambda \in \Lambda(k, n)\right\}$$

where $V(f^{\lambda})$ is the Geöcze area of $f^{\lambda}$.

In [G, Theorem 1] it was shown that if the infimum of the numbers $\delta(P)$, taken over all $P$ as above, is zero, then the Cesari-Weierstrass integral

$$T(g)(\phi) = \lim_{\delta(P) \to 0} \sum_{\pi \in P} \frac{1}{|\pi|} \int_\pi g \circ m(x) \phi(f(x)) \cdot u(f, \pi) \, dx$$

exists whenever $g \in C(M)$ and $\phi$ is an infinitely differentiable $k$-form on $\mathbb{R}^n$. Here $|\pi|$ denotes the $k$ dimensional Lebesgue measure of $\pi$. The current valued linear mapping $T$ on $C(M)$ possesses a unique extension to the class of all bounded Borel measurable functions on $M$ such that Lebesgue’s bounded convergence theorem holds.
Theorem. If the k dimensional Lebesgue measure of \( p^k \circ l(C^k) \) is zero for each \( \lambda \in \Lambda(k, n) \), then:

1. A current valued measure \( T \) can be associated with \( f \) by means of the Cesari-Weierstrass integral.

2. If \( \{f_i\} \) is any sequence of quasi-linear mappings \( f_i : X \to R^k \) converging uniformly to \( f \) with bounded areas, then

\[
T(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ f_i^\#) \phi
\]

whenever \( g \in C_0(M) \) and \( \phi \in E^k(R^k) \).

3. The total variation measure \( \|T\| \) of \( T \), taken with respect to mass, coincides with the Göcze area measure over \( M \).

The notation is that of \([F1]\) and \([G]\).

In case either \( k = 2 \) or the \( k+1 \) dimensional Hausdorff measure of \( f(X) \) is zero, the conclusions of the Theorem above can be obtained using the results of \([F1]\) and \([G]\). It is the purpose of this note to show that the given weaker hypothesis suffices; in particular that one can avoid the use of \([G, \text{Theorem 5}]\).

Conclusion (2) of the Theorem should be compared with \([F1, 3.10]\). Note that \( f|X \setminus m^{-1}(\text{Bdry } X) \) need not be proper.

2. The case \( k = n \). Suppose \( f \) is a continuous mapping with finite Lebesgue area from a polyhedral region \( X \subseteq R^k \) into \( R^k, 2 \leq k \). By \([G, \text{Theorems 1 and 2}]\) applied with \( k = n \), the current valued measure \( T \) associated with \( f \) by means of the Cesari-Weierstrass integral exists and the total variation measure \( \|T\| \) of \( T \) coincides with the Göcze area measure over \( M \).

The arguments of \([F1, 3.4 \text{ and } 3.6]\) show that there exists a unique current valued measure \( S \) over \( M \setminus m(\text{Bdry } X) \) such that, for every sequence \( \{f_i\} \) of quasi-linear maps \( f_i : X \to R^k \) converging uniformly to \( f \) with bounded areas, we have

\[
S(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ f_i^\#) \phi
\]

for \( g \in C_0(M) \) and \( \phi \in E^k(R^k) \).

Since the Lebesgue area of \( f \) and the Göcze area of \( f \) are equal when \( k = n \), there exists a sequence \( \{f_i\} \) of quasi-linear maps converging uniformly to \( f \) with areas converging to the Göcze area of \( f \). Application of \([G, \text{Theorem 4}]\) to this case shows that

\[
T(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ f_i^\#) \phi
\]

for \( g \in C_0(M) \) and \( \phi \in E^k(R^k) \).
3. Proof of the Theorem. Suppose \( \{f_i\} \) is any sequence of quasi-linear maps converging uniformly to \( f \) with bounded areas. As in [F1, 3.4] there is a subsequence, which we continue to denote by \( \{f_i\} \), such that

\[
S(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ m_i) f_i^\# \phi
\]

exists whenever \( g \in C(M) \) and \( \phi \in E^k(\mathbb{R}^n) \). The current valued mapping \( S \) so defined possesses a unique extension, which we continue to denote by \( S \), to the class of all bounded Borel measurable functions on \( M \) such that Lebesgue's bounded convergence theorem holds. With obvious modifications, the argument of [F1, 3.4] shows that \( S(B) \) is a rectifiable current whenever \( B \) is a Borel subset of \( M \setminus m(\text{Bdry } X) \).

From [N1] we see that the infimum of the numbers \( \delta(P) \), taken over all finite collections \( P \) of nonoverlapping simple polyhedral regions \( \pi \subseteq X \), is zero and hence we can define [G, Theorem 1] a current valued function \( T \) on \( C(M) \) by letting

\[
T(g)(\phi) = \lim_{\delta(P) \to 0} \sum_{\pi \in P} \frac{1}{|\pi|} \int g \circ m(\pi) \phi(f(\pi)) \cdot u(f, \pi) \, dx
\]

whenever \( g \in C(M) \) and \( \phi \in E^k(\mathbb{R}^n) \). We denote by \( T \) also the unique extension of this current valued mapping to the class of all bounded Borel measurable functions on \( M \) such that Lebesgue's bounded convergence theorem holds.

Let \( e_1, \ldots, e_n \) denote the usual basis for the space of covectors in \( \mathbb{R}^n \) and let \( e^\lambda = e^{k_1} \wedge \cdots \wedge e^{k_s}, \lambda \in \Lambda(k, n) \), denote the corresponding basis for the space of \( k \)-covectors in \( \mathbb{R}^n \).

Let \( u \in E^0(\mathbb{R}^n) \) be such that \( u(x) = 1 \) for \( x \in \mathbb{R}^n \) and, for \( \lambda \in \Lambda(k, n) \), consider the signed Borel measures defined over \( M \) by letting

\[
S^\lambda(B) = S(B) \wedge e^\lambda(u) \quad \text{and} \quad T^\lambda(B) = T(B) \wedge e^\lambda(u)
\]

whenever \( B \) is a Borel subset of \( M \).

In order to prove statement (2) it suffices to show that \( S^\lambda(B) = T^\lambda(B) \) whenever \( B \) is a Borel subset of \( M \setminus m(\text{Bdry } X) \).

For fixed \( \lambda \in \Lambda(k, n) \) let \( l_\lambda \circ m_\lambda \) denote the monotone-light factorization of \( f^\lambda = p^\lambda \circ f \) with middle space \( M_\lambda \) and let \( h \) be the monotone map such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{m} & M \\
m_\lambda \downarrow & & \downarrow l_\lambda \\
M_\lambda & \xrightarrow{h} & R^k \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & \xrightarrow{p^\lambda} \\
 & & \downarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & \\
R^n & \xrightarrow{I} & R^k \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & \\
 & & \\
\end{array}
\]
From the case $k=n$, previously considered, we have $S^k(\psi \circ h) = T^k(\psi \circ h)$ whenever $\psi \in C_0(M)$. By Lebesgue’s bounded convergence theorem the above equality holds for any bounded Borel measurable function $\psi$ on $M$ that vanishes on $m(Bdry X)$. In particular

$$S^k(h^{-1}(B)) = T^k(h^{-1}(B))$$

whenever $B$ is a Borel subset of $M \setminus m(Bdry X)$.

For any Borel set $A \subset M \setminus [C_{p\lambda} \cup m(Bdry X)]$, we have

$$h(A) \subset M \setminus m(Bdry X) \quad \text{and} \quad h^{-1}(h(A)) = A.$$ 

Hence $S^k(A) = T^k(A)$.

We will complete the proof of statement (2) by showing that $T^k(B)$ and $S^k(B)$ both vanish for any Borel set $B \subset C_{p\lambda} \setminus m(Bdry X)$.

Let $\mu_\lambda$ denote the finite Borel measure $[N2, 6.10]$ defined over $M_\lambda$ by letting

$$\mu_\lambda(B) = \inf \{ V(f^k | m_\lambda^{-k}(U)) : U \text{ open in } M_\lambda, U \supset B \}$$

whenever $B$ is a Borel subset of $M_\lambda$. Here $V(f^k | m_\lambda^{-k}(U))$ denotes the Geöcze area of $f^k | m_\lambda^{-k}(U)$.

For $\eta > 0$ let $C^\eta = M \cap \{ z : \text{diam } h^{-1}(h(z)) \geq \eta \}$. Then each $C^\eta$ is closed and $C_{p\lambda} = \bigcup_{\eta > 0} C^\eta$.

Suppose $\eta > 0$ and $P$ is a finite collection of nonoverlapping simple polyhedral regions $\pi \subset X$ with $\delta(P) < \eta$. Then, for each $\pi \in P$, we have

$$m_\lambda(\pi) \cap h(C^\eta) \subset m_\lambda(Bdry \pi).$$

By $[N2, 6.5]$, $V(f^k | \pi) = \mu_\lambda(m_\lambda(\pi) \setminus m_\lambda(Bdry \pi))$ and hence

$$\mu(M_\lambda) = V(f^k) \leq \sum_{\pi \in P} V(f^k | \pi) + \delta(P)$$

$$= \sum_{\pi \in P} \mu_\lambda(m_\lambda(\pi) \setminus m_\lambda(Bdry \pi)) + \delta(P)$$

$$\leq \mu_\lambda(M_\lambda \setminus h(C^\eta)) + \delta(P).$$

Thus $\mu_\lambda(h(C^\eta)) = 0$ for each $\eta > 0$ and hence $\mu_\lambda(h(C_{p\lambda})) = 0$.

If $U$ is open in $M_\lambda$, then $h^{-1}(U)$ is open in $M$ and $m_\lambda^{-1}(U) = m^{-1}(h^{-1}(U))$. Thus

$$\inf \{ V(f^k | m^{-k}(U)) : U \text{ open in } M, U \supset C_{p\lambda} \} = 0.$$ 

(This argument was suggested by T. Nishiura $[N3, 4.5]$.) Since

$$T^k(\psi) = \lim_{\delta(P) \to 0} \sum_{\pi \in P} \frac{1}{\delta(P)} \int_\pi \psi \circ m(x) u(f^k, \pi) \, dx$$

for $\psi \in C(M)$, we infer that $T^k(B) = 0$ whenever $B$ is a Borel subset of $C_{p\lambda}$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
If $B$ is a Borel subset of $C^p \setminus m(B_{\text{dry}} X)$ whose closure is contained in $C^p$, then $S(B)$ is rectifiable and $\text{spt } S(B) \subset l(C^p)$. Thus, by [F2, 2.1], $S^4(B) = 0$. Since $C^p$ is an $F_\sigma$ set, we have $S^4(B) = 0$ for each Borel set $B \subset C^p \setminus m(B_{\text{dry}} X)$ and statement (2) follows.

Statement (3) now follows from [G, Theorems 3 and 7].

4. Example. We consider here an example from [N1, 2.2] to show that, in case $2 < k < n$, the hypothesis of the Theorem may be satisfied even though the $k + 1$ dimensional measure of $f(X)$ is positive.

Let $X_0$ denote the unit square in $R^2$ and let $f$ denote a continuous mapping with finite area from $X_0$ into $R^3$ such that $\bar{f}(X_0)$ has positive $3$ dimensional measure. If $\bar{p}: R^3 \to R^2$ is any orthogonal projection and $l$ is the light factor of $\bar{f}$, then the $2$ dimensional measure of $\bar{p} \circ l(C_3)$ is zero.

Let $X = X_0 \times I$ where $I$ is the unit interval and let $f$ denote the continuous mapping with finite Lebesgue area from $X$ into $R^4$ defined by letting

$$f(x) = (\bar{f}(x_1, x_2), x_3) \quad \text{for } x = (x_1, x_2, x_3) \in X.$$ 

The $4$ dimensional measure of $f(X)$ is positive. If $p: R^4 \to R^3$ is an orthogonal projection such that $p(e_4) = 0$ and $l$ is the light factor of $f$, then the $3$ dimensional measure of $p \circ l(C_3)$ is equal to that of $\bar{f}(X_0)$ and hence is positive. If, however, $p(e_4) \neq 0$, then it is readily seen that the $3$ dimensional measure of $p \circ l(C_3)$ is zero.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506