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COMPLETELY POSITIVE MAPS ON U^*-ALGEBRAS

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ABSTRACT. Completely positive maps on U^*-algebras with identity are characterized in terms of *-representations on Hilbert space. A result on restricted multiplicativity of such maps is established, from which it follows that completely positive maps which take unitaries to unitaries are *-homomorphisms. It is also shown that positive maps on commutative U^*-algebras with identity are completely positive.

Banach algebras with involution may be profitably studied in the purely algebraic setting of U^*-algebras, recently introduced by Palmer. In this paper we give U^*-algebra versions of two results of Stinespring concerning completely positive linear maps of B^*-algebras into the algebra of bounded operators on a Hilbert space. We also show that if φ is a 1-preserving completely positive map on a U^*-algebra A with identity, then the set $\{x \in A : \varphi(x^*x) = \varphi(x)^*\varphi(x)\}$ is a subalgebra of A on which φ is multiplicative.

1. **Preliminaries.** All algebras and spaces considered here will be over the complex field C. An algebra A with involution $a \to a^*$ (henceforth called a *-algebra) is a *U^*-algebra* if it is the span of the set $\{v \in A : v+v^* = v^*v = vv^*\}$. Elements of this set are called *quasi-unitary*. If A is a *-algebra with multiplicative identity 1, then $u \in A$ is *unitary* ($uu^* = u^*u = 1$) iff $1-u$ is quasi-unitary, so a U^*-algebra with 1 is spanned by its unitaries. All Banach *-algebras are U^*-algebras. For a summary of important results on U^*-algebras, see [**2**]; details may be found in [**3**, §2.2].

Let $\mathfrak{H}$ be a Hilbert space, $B(\mathfrak{H})$ the algebra of bounded operators on $\mathfrak{H}$. A linear map φ from a *-algebra A into $B(\mathfrak{H})$ is called *positive* if $\varphi(a^*a)$ is a positive operator for all $a \in A$. We call φ *completely positive* if for every $n \geqq 1$ and every $a_1, \cdots, a_n \in A$, $\xi_1, \cdots, \xi_n \in \mathfrak{H}$, we have

$$\sum_{i,j} (\varphi(a_i^*a_j)\xi_j, \xi_i) \geqq 0.$$

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Let $A_{(n)}$ denote the algebra of $n\times n$ matrices with entries from A, made into a *-algebra by defining $[a_{ij}]^*=[a^*_{ji}]$ for matrices $[a_{ij}]\in A_{(n)}$. We get a map φ_n from $A_{(n)}$ into $B(\mathfrak{H})_{(n)}$ (regarded as the algebra of bounded operators on the Hilbert space direct sum of n copies of $\mathfrak{H}$) by setting $\varphi_n([a_{ij}])=[\varphi(a_{ij})]$ for each $[a_{ij}]\in A_{(n)}$. Easy computations show that φ is completely positive iff φ_n is positive for all $n\geqq 1$. Complete positivity is a stronger notion than positivity; for example, the map which sends each 2×2 matrix to its transpose is positive but not completely positive. Completely positive maps on B^*-algebras are discussed in **[1]** and **[5]**.

2. **Characterization of complete positivity.** If A is a *-algebra, π a *-representation of A on a Hilbert space $\mathfrak{K}$, and V a bounded linear map from another Hilbert space $\mathfrak{H}$ into $\mathfrak{K}$, then a direct computation shows that the map $\varphi:A\to B(\mathfrak{H})$ defined by $\varphi(a)=V^*\pi(a)V$ for $a\in A$ is completely positive. Stinespring **[5]** showed that all completely positive maps from a B^*-algebra with 1 have this form. Arveson's slightly simplified proof of this result in **[1]** can be modified to give a characterization of completely positive maps on U^*-algebras with 1.

THEOREM 1. *Let A be a U^*-algebra with 1, $\mathfrak{H}$ a Hilbert space, and $\varphi:A\to B(\mathfrak{H})$ a completely positive map. There is a Hilbert space $\mathfrak{K}$, a *-representation π of A on $\mathfrak{K}$, and a bounded linear map $V:\mathfrak{H}\to\mathfrak{K}$ such that $\varphi(a)=V^*\pi(a)V$ for all $a\in A$.*

PROOF. For each $\xi\in\mathfrak{H}$, the map $a\to(\varphi(a)\xi,\xi)$ is a positive linear functional on A. Since positive functionals on a *-algebra with 1 are automatically hermitian **[4]**, we have $(\varphi(a)\xi,\xi)=(\xi,\varphi(a^*)\xi)$ for all $a\in A$ and hence $\varphi(a^*)=\varphi(a)^*$.

Let $\mathfrak{K}_0$ be the algebraic tensor product $A\otimes\mathfrak{H}$. Define a map $(\cdot,\cdot)$ from $\mathfrak{K}_0\times\mathfrak{K}_0$ to C by

$$\left(\sum_{i=1}^{n} a_i\otimes\xi_i,\ \sum_{j=1}^{m} b_j\otimes\eta_j\right)=\sum_{i,j}(\varphi(b_j^*a_i)\xi_i,\eta_j).$$

Then $(x,x)\geqq 0$ for all $x\in\mathfrak{K}_0$ (since φ is completely positive) and $(x,y)=\overline{(y,x)}$ for all $x,y\in\mathfrak{K}_0$ (since φ is a *-map), so $(\cdot,\cdot)$ is a semi-inner product on $\mathfrak{K}_0$. Make $\mathfrak{K}_0$ into a left A-module by defining

$$a\cdot\left(\sum_{i=1}^{n} a_i\otimes\xi_i\right)=\sum_{i=1}^{n} aa_i\otimes\xi_i.$$

Observe that if $u\in A$ is unitary, then $(u\cdot x,u\cdot x)=(x,x)$ for all $x\in\mathfrak{K}_0$.

Let $\mathfrak{N}_0=\{x\in\mathfrak{K}_0:(x,x)=0\}$. The usual arguments show that $\mathfrak{N}_0$ is a linear subspace of $\mathfrak{K}_0$ and $\mathfrak{K}_0/\mathfrak{N}_0$ is a pre-Hilbert space in the inner product $(x+\mathfrak{N}_0,y+\mathfrak{N}_0)=(x,y)$. If $u\in A$ is unitary and $x\in\mathfrak{N}_0$, then $u\cdot x\in\mathfrak{N}_0$.

Since A is spanned by its unitaries, this is enough to show that $a\cdot\mathfrak{N}_0\subseteq\mathfrak{N}_0$ for all $a\in A$. For each $a\in A$, we can thus define a linear map $\pi_0(a)$ on $\mathfrak{K}_0/\mathfrak{N}_0$ by $\pi_0(a)(x+\mathfrak{N}_0)=a\cdot x+\mathfrak{N}_0$. For $x, y\in\mathfrak{K}_0$ and $a\in A$, we have $(\pi_0(a)(x+\mathfrak{N}_0), y+\mathfrak{N}_0)=(a\cdot x, y)=(x, a^*\cdot y)=(x+\mathfrak{N}_0, \pi_0(a^*)(y+\mathfrak{N}_0))$, so π_0 is a *-representation of A on $\mathfrak{K}_0/\mathfrak{N}_0$. If $u\in A$ is unitary, then $\pi_0(u)$ is an isometry of $\mathfrak{K}_0/\mathfrak{N}_0$. Each $\pi_0(a)$, being a linear combination of isometries, is therefore continuous with respect to the inner product on $\mathfrak{K}_0/\mathfrak{N}_0$ and extends uniquely to a bounded operator $\pi(a)$ on the Hilbert space completion $\mathfrak{K}$ of $\mathfrak{K}_0/\mathfrak{N}_0$. π is a *-representation of A on $\mathfrak{K}$.

Define $V:\mathfrak{H}\to\mathfrak{K}$ by $V\xi=1\otimes\xi+\mathfrak{N}_0$ for $\xi\in\mathfrak{H}$. We have $\|V\xi\|^2=(1\otimes\xi, 1\otimes\xi)=(\varphi(1)\xi, \xi)\leqq\|\varphi(1)\|\,\|\xi\|^2$, so V is bounded. For ξ, $\eta\in\mathfrak{H}$ and $a\in A$, $(V^*\pi(a)V\xi, \eta)=(\pi(a)V\xi, V\eta)=(a\otimes\xi, 1\otimes\eta)=(\varphi(a)\xi, \eta)$ so $\varphi(a)=V^*\pi(a)V$ as desired.

3. **Multiplicativity of completely positive maps.**

THEOREM 2. *Let A be a U^*-algebra with 1, $\mathfrak{H}$ a Hilbert space, and $\varphi:A\to B(\mathfrak{H})$ a completely positive map such that $\varphi(1)=I_{\mathfrak{H}}$ (identity operator on $\mathfrak{H}$). If $x\in A$ satisfies $\varphi(x^*x)=\varphi(x)^*\varphi(x)$, then $\varphi(yx)=\varphi(y)\varphi(x)$ for all $y\in A$. Hence $\{x\in A:\varphi(x^*x)=\varphi(x)^*\varphi(x)\}$ is a subalgebra of A on which φ is multiplicative.*

PROOF. Let $\mathfrak{K}$, π, and V be as in Theorem 1. We have $V^*\pi(x)^*\pi(x)V=V^*\pi(x)^*VV^*\pi(x)V$ by assumption. Let $T=(I_{\mathfrak{K}}-VV^*)\pi(x)VV^*$. Since $\varphi(1)=V^*V=I_{\mathfrak{H}}$, VV^* is a projection in $B(\mathfrak{K})$. A direct computation now shows that $T^*T=0$, so $T=0$, so $\pi(x)VV^*=VV^*\pi(x)VV^*$. For any $y\in A$ we have $V\varphi(yx)V^*=VV^*\pi(y)\pi(x)VV^*=VV^*\pi(y)VV^*\pi(x)VV^*=V\varphi(y)\varphi(x)V^*$. Multiply on the left by V^*, and on the right by V to see that $\varphi(yx)=\varphi(y)\varphi(x)$.

COROLLARY. *If $\varphi:A\to B(\mathfrak{H})$ is completely positive and takes unitaries to unitaries, then φ is a *-homomorphism.*

PROOF. Observe that $\varphi(1)$, being positive and unitary, must be $I_{\mathfrak{H}}$, and apply the theorem to see that $\varphi(yu)=\varphi(y)\varphi(u)$ for each $y\in A$ and each unitary $u\in A$. Since A is spanned by its unitaries, φ is multiplicative on A. It has already been noted that φ is a *-map, so φ is a *-homomorphism.

We remark that the above corollary also follows from the proof of Theorem 2.2.7 in [**1**].

4. **Factorization through B^*-algebras.** The *Gel'fand-Naimark pseudonorm* γ on a U^*-algebra A is defined by

$$\gamma(a) = \sup\{\|\pi(a)\| : \pi \text{ a *-representation of } A \text{ on Hilbert space}\}$$

for $a \in A$. Since A is the span of its quasi-unitaries, it is clear that $\gamma(a)$ is finite for each $a \in A$. γ is an algebra pseudonorm A and satisfies the B^* condition $\gamma(a^*a)=\gamma(a)^2$ for all $a \in A$. Let A_R be the intersection of all kernels of *-representations of A on Hilbert space, the so-called *-radical of A. γ vanishes precisely on A_R, so A/A_R is a normed *-algebra in the norm induced by γ. Let B be the completion of A/A_R in this norm. B is then a B^*-algebra. Let ψ denote the natural *-homomorphism of A into B.

PROPOSITION. *Let A be a U^*-algebra with* 1, $\mathfrak{H}$ *a Hilbert space*, $\varphi: A \to B(\mathfrak{H})$ *a positive map. There is a positive map* $\tilde{\varphi}: B \to B(\mathfrak{H})$ *such that* $\varphi = \tilde{\varphi} \circ \psi$.

PROOF. We first show that φ is continuous with respect to the pseudonorm γ. For each unit vector $\xi \in \mathfrak{H}$, we get a positive linear functional f_ξ on A defined by $f_\xi(a)=(\varphi(a)\xi, \xi)$. Each f_ξ in turn gives rise to a *-representation π_ξ of A on a Hilbert space $\mathfrak{H}_\xi$ and a vector $\eta_\xi \in \mathfrak{H}_\xi$ such that $f_\xi(a)=(\pi_\xi(a)\eta_\xi, \eta_\xi)$ for all $a \in A$ (see [2]). We have $\|\eta_\xi\|^2 = f_\xi(1)=(\varphi(1)\xi, \xi) \leqq \|\varphi(1)\|$ and hence $|f_\xi(a)| \leqq \|\pi_\xi(a)\| \|\eta_\xi\|^2 \leqq \gamma(a)\|\varphi(1)\|$ for all unit vectors $\xi \in \mathfrak{H}$ and all $a \in A$. For any $T \in B(\mathfrak{H})$, we have

$$\|T\| \leqq 2 \sup\{|(T\xi, \xi)| : \xi \in \mathfrak{H}, \|\xi\| = 1\}$$

so $\|\varphi(a)\| \leqq 2\|\varphi(1)\|\gamma(a)$, i.e. φ is continuous with respect to γ.

We can now "factor" φ through B. Since φ vanishes on A_R, it induces a positive map $\tilde{\varphi}: A/A_R \to B(\mathfrak{H})$. $\tilde{\varphi}$ is continuous with respect to the norm on A/A_R induced by γ and hence extends to a positive map on B. We have $\varphi = \tilde{\varphi} \circ \psi$.

5. **Commutative U^*-algebras.** Stinespring [5] shows that every positive map from a commutative B^*-algebra into $B(\mathfrak{H})$ is completely positive. The proof of this result relies heavily on realizing such an algebra as the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space, and so cannot be generalized to U^*-algebras directly. By using the factorization developed above, however, we may apply Stinespring's result to get the following theorem.

THEOREM 3. *Every positive map from a commutative U^*-algebra with* 1 *into $B(\mathfrak{H})$ is completely positive.*

PROOF. A positive map φ from a U^*-algebra A with 1 into $B(\mathfrak{H})$ can be factored as $\varphi = \tilde{\varphi} \circ \psi$, where $\tilde{\varphi}: B \to B(\mathfrak{H})$ is positive. If A is commutative then so is B, and Stinespring's result says that $\tilde{\varphi}$ is completely positive. The complete positivity of φ is now immediate.

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