A CHARACTERIZATION OF SPACES ON WHICH ALL PATH MAPS ARE CONTINUOUS

P. W. HARLEY III

Abstract. Here it is shown that all path maps on a topological space $X$ are continuous if and only if $X$ is a quotient image of a locally path connected metric space. This characterization strengthens results of E. Bedford [1].

1. Introduction. A function $f: X \rightarrow Y$ between the topological spaces $X$ and $Y$ is a path map if, whenever $\sigma: I \rightarrow X$ ($I = [0, 1]$) is a path in $X$, so is $f \sigma: I \rightarrow Y$ a path in $Y$. E. Bedford studied path maps in [1] and established the following theorem.

Theorem 1.1. If $X$ is Hausdorff and first countable then every path map on $X$ is continuous if and only if $X$ is locally path connected.

Using different methods we have been able to give a characterization (reminiscent of S. P. Franklin's characterization of sequential spaces [3]; see also [4]) of spaces on which all path maps are continuous as quotient images of locally path connected metric spaces. Thus we have proved:

Theorem 1.2. All path maps on a topological space $X$ are continuous if and only if $X$ is a quotient image of a locally path connected metric space.

The extent to which this theorem strengthens Bedford's result is illustrated by the following corollary.

Corollary 1.3. If all path maps on a topological space $X$ are continuous, then $X$ is locally path connected.

This corollary removes the Hausdorff and first countable hypotheses from one half of Theorem 1.1. A slight modification of Bedford's proof (put $f_n = n$ if it does not exist as defined) will remove the Hausdorff assumption from the converse, but without first countability the converse is false.

Example 1.4. Let $X$ denote the space of all continuous functions $[0, 1] \rightarrow [0, 1]$ with the point-open topology.

Received by the editors October 11, 1971.

AMS 1969 subject classifications. Primary 5425, 5435, 5460.

Key words and phrases. Locally path connected, path maps, metric space, quotient maps.

© American Mathematical Society 1972
It is clear that $X$ is Hausdorff and locally path connected. As an example of a discontinuous path map on $X$ we offer the ordinary Riemann integral $J$. By dominated convergence, $J$ is sequentially continuous. Thus if $\sigma : I \to X$ is a path, $J\sigma$ is sequentially continuous, thus a path.

2. Proofs. Recall that $X$ is locally path connected if it has a basis of path connected open sets.

Proof of Theorem 1.2. Let all path maps on $X$ be continuous. Denote by $X^I$ the set of all paths $I \to X$ and by $Z$ the product set $X^I \times I$. Define a metric $d$ on $Z$ by

$$d((\sigma, t), (\sigma', t')) = \begin{cases} 1, & \text{if } \sigma \neq \sigma', \\ |t - t'|, & \text{if } \sigma = \sigma'. \end{cases}$$

With this metric $Z$ becomes the topological sum of the spaces $\{\sigma\} \times I$, where $\sigma : I \to X$ is a path. Consequently, $Z$ is locally path connected, and the (onto) function $f : Z \to X$ defined by $f(\sigma, t) = \sigma(t)$ is continuous. Hence, the proof will be completed if $f$ is shown to be a quotient. To this end let $g : X \to Y$ be any function for which the composition $gf$ is continuous. Then, by construction, $g$ is easily seen to be a path map on $X$, thus continuous. Therefore, by [2, p. 123], $f$ is a quotient map.

Conversely, suppose that $Z$ is a locally path connected metric space and $f : Z \to X$ a quotient map. If $g : X \to Y$ is a path map on $X$, then $gf : Z \to Y$ is a path map on $Z$, thus continuous by Theorem 1.1. But since $f$ is a quotient map, $g$ must also be continuous.

Proof of Corollary 1.3. If all path maps on $X$ are continuous, represent $X$ as a quotient image of a locally path connected metric space $Z$. The result now follows from the preservation of local path connectedness by quotient maps.

References