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PERMANENT GROUPS

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ABSTRACT. A permanent group is a group of nonsingular matrices on which the permanent function is multiplicative. Let $A \circ B$ denote the Hadamard product of matrices A and B. The set of groups G of nonsingular $n \times n$ matrices which contain the diagonal group $\mathscr{D}$ and such that for every pair A, B of matrices in G we have $A \circ B^T \in \mathscr{D}$ is denoted by $\mathscr{A}_n$.

If the underlying field has at least three elements then $\mathscr{A}_n$ consists of permanent groups. A partial converse is available:

If a permanent group G is generated by $\mathscr{D}$ together with a set S of elementary matrices and a set Q of permutation matrices then $G=HK$ where H is the subgroup generated by Q and K is generated by $\mathscr{D}$ and S, and $K \in \mathscr{A}_n$.

1. **Introduction.** A group of nonsingular matrices on which the permanent function is multiplicative will be called a permanent group. In this paper we determine a large class of permanent groups.

It was conjectured by Marcus and Minc [2] that Δ_n, the group of $n \times n$ nonsingular matrices of the form PD is a maximal permanent group, where P is any permutation matrix and D is diagonal. The underlying field was unspecified in [2] and the conjecture was subsequently verified by the first author for the complex numbers [1]. Contrary to the assertion in [1], however, this is not the only maximal permanent group.

Let $A \circ B$ denote the Hadamard product of matrices A and B. We shall denote by $\mathscr{A}_n(F)$ the collection of all groups G of $n \times n$ nonsingular matrices over a field F such that:

(a) G contains the set of all $n \times n$ nonsingular diagonal matrices;

(b) if A and B are in G then $A \circ B^T$ is a nonsingular diagonal matrix.

In Theorem 3.1 we will show that $\mathscr{A}_n(F)$ consists of permanent groups if F has more than 2 elements. However, Δ_n is not in $\mathscr{A}_n(F)$.

The matrix E_{ij} has (i,j) entry 1 and zeros elsewhere. We will use elementary matrices $I+\lambda E_{ij}$, where λ is a field element, I the identity matrix, and $i \neq j$ for $i,j=1, \cdots, n$. For any matrix A, $r_i(A)$ will denote

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the number of nonzero elements in row i. If row i and column i are deleted from a matrix A the resulting matrix is denoted by $A(i|i)$.

2. **Properties of $\mathscr{A}_n(F)$.** We remark that $\mathscr{A}_n(F)$ contains both the permanent group of nonsingular upper triangular matrices and the permanent group of nonsingular lower triangular matrices. A less obvious permanent group is obtained by considering $n\times n$ upper triangular matrices (a_{ij}) over any field which satisfy the additional restrictions:

$$\begin{aligned} a_{\alpha m} &= 0, \qquad m = \alpha+1, \cdots, \beta, \\ a_{m\beta} &= 0, \qquad m = \alpha+1, \cdots, \beta-1, \end{aligned} \tag{2.1}$$

for distinct integers α, β such that $1\leqq\alpha<\beta\leqq n$.

If P is the permutation matrix corresponding to the transposition $(\alpha\beta)$ and H the group $\{I, P\}$ then the set $H\cdot K$ can be shown to be a permanent group, where K consists of those nonsingular upper triangular matrices satisfying (2.1).

The following two lemmas will be used in proving that $\mathscr{A}_n(F)$ contains only permanent groups.

2.1 Lemma. *If $A\in G\in\mathscr{A}_n(F)$, then $a_{ii}\neq 0$ for $i=1,\cdots,n$.*

Proof. Suppose $A\in G\in\mathscr{A}_n(F)$, but $a_{ii}=0$ for some $i=1,\cdots,n$. Since A is nonsingular there exists $j=1,\cdots,n$, such that $j\neq i$ and both a_{ij} and its cofactor A_{ij} are nonzero. Consequently, the matrix $A^{-1}\in G$ has a nonzero (j, i) entry, contradicting $G\in\mathscr{A}_n(F)$.

2.2 Lemma. *If $G\in\mathscr{A}_n(F)$ and F has at least 3 elements then there exists a pair of positive integers i, j $(1\leqq i, j\leqq n)$ such that, for any $A\in G$,*

$$\begin{aligned} a_{im} &= 0, \qquad m\neq i, \\ a_{mj} &= 0, \qquad m\neq j. \end{aligned} \tag{2.2}$$

Proof. The result is immediate when $n=2$ and the proof proceeds by induction on n.

Let S be the set of all i $(1\leqq i\leqq n)$ for which there exists a matrix $A\in G$ such that $a_{1i}\neq 0$. Always, $1\in S$. We shall show there is a j $(1\leqq j\leqq n)$ such that if B is any matrix in G, then $b_{mj}=0$ for all $m\neq j$. If $S=\{1,\cdots,n\}$, then $G\in\mathscr{A}_n(F)$ implies j may be chosen as 1. Otherwise, suppose B is a matrix in G such that $b_{1i}\neq 0$, and let $D=\operatorname{diag}[1,\cdots,x,\cdots,1]$ where x is the (i, i) entry. Consider an arbitrary matrix C in G. Then BDC is in G and its $(1, j)$ entry is

$$\sum_{k=1;k\neq i}^{n} b_{1k}c_{kj} + b_{1i}c_{ij}x. \tag{2.3}$$

If c_{ij} is nonzero, then (2.3) vanishes for at least one x in F. Since F has at least 3 elements we may always choose a nonzero x in F such that (2.3) is also nonzero. This implies that $c_{ij}=0$ whenever $i \in S$ and $j \in S'$, the complement of S in $\{1, \cdots, n\}$.

Since C was arbitrary in G there is a permutation matrix P such that, for any $A \in G$,

$$P^{-1}AP = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

where A_1 is $k \times k$ (k is the cardinality of S').

The set $\{P^{-1}AP | A \in G\}$ is also a group and hence $\{A_1 | A \in G\} = G_1$ is a group of $k \times k$ matrices. Evidently $G_1 \in \mathscr{A}_k(F)$. Hence, by induction, each $A_1 \in G_1$ satisfies (2.2) for $m=1, \cdots, k$. Hence $P^{-1}AP$ satisfies

$$(P^{-1}AP)_{mj} = 0, \qquad m \neq j, m = 1, \cdots, n.$$

Thus, there exists j', not necessarily j, such that

$$a_{mj'} = 0, \qquad m \neq j', m = 1, \cdots, n.$$

A similar argument yields an i' such that

$$a_{i'm} = 0, \qquad m \neq i', m = 1, \cdots, n.$$

3. The main theorem.

3.1 THEOREM. *If F is a field with at least* 3 *elements, then every group in each $\mathscr{A}_n(F)$ is a permanent group.*

This is an immediate consequence of the following two lemmas:

3.2 LEMMA. *If $A, B \in G \in \mathscr{A}_n(F)$, then the ith diagonal entry of AB is $a_{ii}b_{ii}$, for all $i=1, \cdots, n$.*

PROOF. The (i, i) entry of AB is $\sum_{k=1}^{n} a_{ik}b_{ki}$. If $k \neq i$ and $a_{ik} \neq 0$, then $b_{ki}=0$ since $B \in G \in \mathscr{A}_n(F)$. On the other hand, if $k \neq i$ and $b_{ki} \neq 0$, then $a_{ik}=0$.

3.3 LEMMA. *If $A \in G \in \mathscr{A}_n(F)$, then* per $A = \prod_{i=1}^{n} a_{ii}$.

PROOF. This is obvious for $n=1, 2$. Assume the result for all $k<n$. By Lemma 2.2 there is a column of A, say the jth, where the only nonzero entry is the diagonal one. It follows that the matrices $A(j|j)$ form a group in $\mathscr{A}_{n-1}(F)$. This group will contain the nonsingular diagonal matrices because G does. By induction, per $A(j|j) = \prod_{i \neq j}^{n} a_{ii}$ and so per $A = a_{jj}$ per $A(j|j) = \prod_{i=1}^{n} a_{ii}$.

4. **A partial converse.** Clearly not every permanent group G is in some $\mathscr{A}_n(F)$, e.g., $\Delta_n \notin \mathscr{A}_n(F)$. But if a permanent group is generated by the diagonal group together with a subset of the elementary matrices and a collection $\mathscr{Q}$ of permutation matrices, then $G=H\cdot K$ where H is the subgroup generated by $\mathscr{Q}$ and $K\in\mathscr{A}_n(F)$ is generated by the diagonal group together with the given set of elementary matrices (Theorem 4.3).

4.1 LEMMA. *If G is a permanent group generated by the diagonal group together with a set of elementary matrices, then for all $A \in G$, $a_{ij}\neq 0$ implies $E_{ij}(\lambda)\in G$ for every $\lambda \in F$.*

PROOF. Write A as $E_1 \cdots E_m D$ where D is diagonal and each E_i is an elementary matrix in G. The only way a_{ij} can be nonzero is if there is a sequence $i=i_0, i_1, \cdots, i_k=j$ where $E_{i_r i_{r+1}}(\lambda)\in G$ for $r=0, 1, \cdots, k-1$. However E_{ij} is the product of the following elementary matrices

$$E_{i_0 i_1}(\lambda)\prod_{r=1}^{k-1} E_{i_r i_{r+1}}(1) E_{i_0 i_1}(-\lambda)\prod_{r=1}^{k-1} E_{i_r i_{r+1}}(-1).$$

The proof is complete because if any group of nonsingular matrices containing the diagonal group also contains an elementary matrix $E_{pq}(\alpha)$ then it contains $E_{pq}(\beta)$ for every β in F.

4.2 LEMMA. *Any permanent group generated by the diagonal group and a set of elementary matrices is in $\mathscr{A}_n(F)$.*

PROOF. Suppose $a_{ij}\neq 0$ for some $A\in G$, where G is a permanent group satisfying the hypotheses. We must show $b_{ji}=0$ for all $B\in G$. If $b_{ji}\neq 0$ for some $B \in G$ then the previous lemma implies both $E_{ij}(\lambda)$ and $E_{ji}(\lambda)$ are in G. However,

$$\operatorname{per}[E_{ij}(1)\cdot E_{ji}(1)] = 3 \quad \text{while} \quad \operatorname{per} E_{ij}(1) = \operatorname{per} E_{ji}(1) = 1,$$

contradicting the assumption that G was a permanent group.

4.3 THEOREM. *If G is a permanent group generated by the diagonal group, a set of elementary matrices S, and a nonempty set of permutation matrices $\mathscr{Q}$ then $G=H\cdot K$ where H is the subgroup generated by $\mathscr{Q}$ and $K\in\mathscr{A}_n(F)$ is generated by the diagonal group and S.*

PROOF. By virtue of Lemma 4.2, K is in $\mathscr{A}_n(F)$, and we need only show that $HK=KH$. If P is any permutation matrix in H and $E\in K$, then $PE=FP$, where $F=PEP^{-1}$ is readily seen to be in K also.

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