ISOMORPHISM TYPES OF INFINITE
SYMMETRIC GRAPHS

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Abstract. Professor Bjarni Jónsson asked about the cardinality of isomorphism types of infinite symmetric graphs of order m, for each infinite cardinal m. We show that there are $2^m$ pairwise non-isomorphic infinite symmetric graphs of order m, for each infinite cardinal m.

A symmetric graph is an ordered pair $(U, F)$ where $F$ is a symmetric relation over the set $U$. The cardinality of $U$ is referred to as the order of the graph. Professor Bjarni Jónsson stated in [2, p. 31] that, as far as we know, the cardinality of the class of all pairwise nonisomorphic infinite symmetric graphs of order m, for each infinite cardinal m, is unknown. Since $F$ is a subset of $U \times U$, it is trivial to see that $2^m$ is an upper bound. In this paper, we shall settle this cardinality question by proving the following theorem.

Theorem. The cardinality of the isomorphism types of infinite symmetric graphs of order m is $2^m$ for each infinite cardinal m.

We base our proof on the result by Professors Comer and LeTourneau [1] that there are $2^m$ pairwise nonisomorphic 1-unary root algebras of order m, for each infinite cardinal m. With each 1-unary root algebra $A = (U, f)$, we associate the symmetric graph $\bar{A} = (U, F)$ where $F = f \cup f^{-1}$. To complete the proof of our theorem, it is therefore sufficient to prove the following lemma.

Lemma. Let $\bar{A} = (U, F)$ and $\bar{B} = (U, G)$ be two symmetric graphs associated with two 1-unary root algebras $A = (U, f)$ and $B = (U, g)$, respectively. If $\bar{A}$ and $\bar{B}$ are two isomorphic symmetric graphs, then $A$ and $B$ are two isomorphic 1-unary root algebras.

Let $a$ be the fixed point of $A = (U, f)$. Note that $a$ is the only fixed point, i.e., it is the only element of $U$ satisfying $f(a) = a$. Let us write $x \rightarrow^f y$ or...
y \rightarrow^f x \text{ if } f(x) = y. 

(1) For each point } x \in U, \text{ there is one and only one arrow leaving } x. 

(2) The loop arrow } a \rightarrow^f a \text{ is the only arrow leaving the fixed point } a. 

(3) For each } x \neq a \in U \text{ there is a least positive integer } n, \text{ depending on } x, \text{ satisfying } f^n(x) = a. \text{ We shall call } n \text{ the height of } x. \text{ Then, we have connecting arrows from } x \text{ to } a \text{ as follows:}

\[
\begin{array}{ccccccc}
& & x & \xrightarrow{f} & f(x) & \xrightarrow{f} & \cdots & \xrightarrow{f} & f^{n-1}(x) & \xrightarrow{f} & f^n(x) = a
\end{array}
\]

with } f^{n-1}(x) \neq f^n(x), \text{ i.e., } f^{n-1}(x) \neq a.

**Proof.** Assume that } \Phi \text{ is a symmetric graph isomorphism from } \tilde{A} = (U, F) \text{ onto } \tilde{B} = (U, G). \text{ We shall show that } \Phi \text{ is a 1-unary root algebra isomorphism from } A = (U, f) \text{ onto } B = (U, g). \text{ Since } \Phi \text{ already is a bijection on } U, \text{ it suffices to show that } g(\Phi(x)) = \Phi(f(x)) \text{ for each } x \in U, \text{ or equivalently,}

\[
\Phi(x) \xrightarrow{g} \Phi(f(x)) \quad \text{for each } x \in U.
\]

In other words, it is sufficient to show that } \Phi \text{ is arrow preserving.

Since } (x, f(x)) \in f \subseteq f \cup f^{-1} = F \text{ and since } \Phi \text{ is a symmetric graph isomorphism from } \tilde{A} = (U, F) \text{ to } \tilde{B} = (U, G), \text{ we have } \Phi(x, f(x)) \in G. \text{ But, then}

\[
(\Phi(x), \Phi(f(x)) = \Phi(x, f(x)) \in G = g \cup g^{-1},
\]

which means

\[
\begin{array}{c}
\Phi(x) \xrightarrow{g} \Phi(f(x)) \\
\Phi(f(x)) \xleftarrow{g} \Phi(x)
\end{array}
\]

in an obvious sense. Note that to prove (*) is to prove that the top arrow of (4) holds.

If } x = a, \text{ then (4) coincides with } \Phi(a) \rightarrow^g \Phi(a) \text{ in either case which shows, first, that (*) is true in case } x = a \text{ and, second, that}

\[
\Phi(a) \text{ is the fixed point, say, } b \text{ of } \tilde{B} = (U, G).
\]

If } x \neq a, \text{ let } n \geq 1 \text{ be the height of } x. \text{ By our observations (3), (4) and (5), we have}

\[
\begin{array}{ccccccc}
\Phi(x) & \xleftarrow{g} & \Phi(f(x)) & \xleftarrow{g} & \Phi(f^2(x)) & \cdots & \xleftarrow{g} & \Phi(f^{n-1}(x)) & \xleftarrow{g} & \Phi(f^n(x)) = b
\end{array}
\]

with } \Phi(f^{n-1}(x)) \neq \Phi(f^n(x)) = b. \text{ (The upper and lower subscripts of } g \text{ are attached only for the convenience of ensuing quotations.) By our early
observation (2) we see that \( g^n \) holds while \( g_n \) does not. Since \( g^n \) already is an arrow leaving the element \( \phi(f^{n-1}(x)) \) in \( B \), there can be no other arrow leaving \( \phi(f^{n-1}(x)) \) by observation (1). Hence, \( g^{n-1} \) must hold, while \( g_{n-1} \) does not. Similarly applying (1) over and over, we shall have

\[
\phi(x) \xrightarrow{g} \phi(f(x)) \xrightarrow{g} \cdots \xrightarrow{g} \phi(f^{n-1}(x)) \xrightarrow{g} \phi(f^n(x)) = b
\]

the first arrow (from the left) of which surely proves (*). This completes a proof of our lemma and, consequently, our theorem.

Recall [1] that each of the aforementioned Comer-LeTourneau 1-unary root algebras has only the trivial automorphism group. From this, the following is immediate.

**Corollary.** There are \( 2^m \) pairwise nonisomorphic infinite symmetric graphs of order \( m \), for each infinite cardinal \( m \), each with only the trivial automorphism group.

**References**


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