OPERATORS ASSOCIATED WITH A PAIR OF NONNEGATIVE MATRICES
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ABSTRACT. Let $A_{m \times n}$, $B_{m \times n}$, $X_{n \times 1}$, and $Y_{m \times 1}$ be matrices whose entries are nonnegative real numbers and suppose that no row of $A$ and no column of $B$ consists entirely of zeroes. Define the operators $U$, $T$ and $T'$ by $(UX)_i = X_i^{-1}$ [or $(UY)_i = Y_i^{-1}$], $T = UB'UA$ and $T' = UAUB'$. $T$ is called irreducible if for no nonempty proper subset $S$ of $\{1, \ldots, n\}$ it is true that $X_i = 0, i \in S$; $X_i \neq 0, i \notin S$, implies $(TX)_i = 0, i \in S$; $(TX)_i \neq 0, i \notin S$. M. V. Menon proved the following Theorem. If $T$ is irreducible, there exist row-stochastic matrices $A_1$ and $A_2$, a positive number $\theta$, and two diagonal matrices $D$ and $E$ with positive main diagonal entries such that $DAE = A_1$ and $\theta DBE = A_2$. Since an analogous theorem holds for $T'$, it is natural to ask if it is possible that $T'$ be irreducible if $T$ is not. It is the intent of this paper to show that $T'$ is irreducible if and only if $T$ is irreducible.

Suppose that each of $m$ and $n$ is a positive integer. Let $A_{m \times n}$ and $B_{m \times n}$ be matrices whose entries are nonnegative real numbers and suppose that no row of $A$ and no column of $B$ consists entirely of zeroes. Let $X_{n \times 1}$ and $Y_{m \times 1}$ be matrices whose entries are taken from the extended real nonnegative numbers. Define the operator $U$ by $(UX)_i = X_i^{-1}$ [or $(UY)_i = Y_i^{-1}$] and let $0^{-1} = \infty, \infty^{-1} = 0$, $\infty + \infty = \infty, 0 \cdot \infty = 0$, and if $a > 0, a \cdot \infty = \infty$ [1]. Define the operators $T$ and $T'$ by $T = UB'UA$ and $T' = UAUB'$ where $B'$ is the transpose of $B$. Clearly

$$(TX)_i = \left(\sum_{j=1}^{m} b_{ij} \left(\sum_{k=1}^{n} a_{jk} X_k\right)^{-1}\right)^{-1}.$$

$T$ is called irreducible if for no nonempty proper subset $S$ of $N = \{1, \ldots, n\}$ is it true that $X_i = 0, i \in S$; $X_i \neq 0, i \notin S$, implies $(TX)_i = 0, i \in S$; $(TX)_i \neq 0, i \notin S$. $T'$ is defined to be irreducible analogously.

M. V. Menon [2] proved the following.

**Theorem 1.** If $T$ is irreducible, then there exist row-stochastic matrices $A_1$ and $A_2$, a positive number $\theta$, and two diagonal matrices $D$ and $E$ with positive main diagonal entries such that $DAE = A_1$ and $\theta DBE = A_2$.

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Since an analogous theorem holds for $T'$, one would hope that $T'$ might be irreducible even if $T$ is not. However, it is the intent of this paper to prove the following result.

**Theorem 2.** $T'$ is irreducible if and only if $T$ is irreducible.

**Proof.** If $T'$ is not irreducible then there is a nonempty proper subset $S$ of $M = \{1, \ldots, m\}$ which has the property that if $Y_{m \times 1}$ is such that $Y_i = 0$, $i \in S$, $Y_i \neq 0$, $i \notin S$, then $(T' Y)_i = 0$, $i \in S$, $(T' Y)_i \neq 0$, $i \notin S$. Let $Z_{n \times 1} = UB'Y$, put $E = \{i \in N: Z_i = \infty\}$, and let $E'$ be the complement of $E$ in $N$.

1. If $i_0 \in S$ then there exists $j_0 \in N$ such that $a_{i_0 j_0} (\sum_{k=1}^{m} b_{k j_0} Y_k)^{-1} = \infty$. Hence $Z_{j_0} = \infty$ and thus $E$ is not null.

2. Let $S'$ be the complement of $S$ in $M$ so that if $i_0 \in S'$, then there exists $j_0 \in N$ such that $\infty > a_{i_0 j_0} (\sum_{k=1}^{m} b_{k j_0} Y_k)^{-1} > 0$. Hence $\infty > Z_{j_0} > 0$ and $E'$ is not null.

3. Let $X_{n \times 1}$ be defined by putting $\infty > X_i > 0$ if $i \in E$ and $X_i = 0$ if $i \in E'$. If $X_{i_0} = 0$ then $\infty > (\sum_{j=1}^{m} b_{i_0 j} (\sum_{k=1}^{m} a_{j k} Z_k)^{-1}) > 0$ and hence there exists $j_0 \in M$ such that $\infty > b_{i_0 j_0} > 0$. Thus $\infty > \sum_{k=1}^{m} a_{i_0 j_0} Z_k = \sum_{k \in E'} a_{i_0 j_0} Z_k + \sum_{k \in E} a_{i_0 j_0} Z_k > 0$ so that $a_{i_0 j_0} = 0$ for $k \in E$. Therefore, if $X_{i_0} = 0$, then $(TX)_{i_0} = 0$.

4. For $i_0 \in E$, put $F = \{j \in M: b_{i_0 j} = 0\}$ and let $F'$ be the complement of $F$ in $M$. Since $\infty > X_{i_0} > 0$ then $(TZ)_{i_0} = \infty$ so that

$$\sum_{j=1}^{m} b_{j i_0} \left( \sum_{k=1}^{n} a_{j k} Z_k \right)^{-1} = \sum_{j \in F'} b_{j i_0} \left( \sum_{k=1}^{n} a_{j k} Z_k \right)^{-1} + \sum_{j \in F} b_{j i_0} \left( \sum_{k=1}^{n} a_{j k} Z_k \right)^{-1} = 0,$$

and hence there exists $j_0 \in F'$ so that $b_{j_0 i_0} \neq 0$. Thus $\sum_{k=1}^{n} a_{i_0 k} Z_k = \infty$ so that there exists $k_0 \in E$ such that $\infty > a_{i_0 k_0} X_{k_0} > 0$. Therefore if $\infty > X_{i_0} > 0$ then $\infty > (TX)_{i_0} > 0$.

It immediately follows from (1), (2), (3), and (4) that $T'$ is irreducible if $T$ is irreducible. A similar argument proves that $T$ is irreducible if $T'$ is irreducible.

**References**


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