PERTURBED ASYMPTOTICALLY STABLE SETS

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Abstract. Perturbations of a dynamical system are defined and the behavior of compact asymptotically stable sets under these perturbations is determined. The occurrence of critical points in a perturbed planar dynamical system is also investigated.

In [1] it is shown that if $C$ is an asymptotically stable cycle of a planar dynamical system $\pi$ and if $\pi_t$ is a net of planar dynamical systems which converges to $\pi$, then there are limit cycles $C_t$ of $\pi_t$ such that $C_t \rightarrow C$. This paper presents a similar result in a more general setting for perturbed asymptotically stable sets. If $\pi_t$ is a net of dynamical systems which converges to a dynamical system $\pi$ and if $M$ is a compact asymptotically stable set of $\pi$, then eventually there are asymptotically stable sets $M_t$ of $\pi_t$ arbitrarily close to $M$. Moreover, if $M$ is invariant with respect to all $\pi_t$, then $M_t \rightarrow M$.

$R$, $R^+$, and $R^-$ will denote the reals, nonnegative reals, and nonpositive reals respectively.

A dynamical system $\pi$ on a topological space $X$ is a mapping of $X \times R$ onto $X$ which satisfies the following three conditions (where $x \pi t = \pi(x, t)$):

(i) $\pi$ is continuous in the product topology.
(ii) $x \pi 0 = x$ for each $x \in X$.
(iii) $(x \pi t) \pi s = x \pi (t+s)$ for each $x \in X$ and $s, t \in R$.

If $A \subset X$ and $B \subset R$, then $A \pi B$ will denote the set $\{x \pi t : x \in A, t \in B\}$. $L^+(x)$ and $L^-(x)$ will denote the positive limit set of $x$ and the negative limit set of $x$ respectively. A subset $M$ of $X$ is called an (negative) attractor iff there is a neighborhood $U$ of $M$ such that $(L^-(x)) \cup U = M$ for every $x \in U$. If $M$ is an (negative) attractor, then $(A^+(M)) \cup A^-(M)$ will denote the largest such neighborhood.

A subset $S$ of $X$ is called a section with respect to $\pi$ iff $(S \pi t) \cap S = \emptyset$ for all $t \neq 0$.

In a topological space $X$ it is possible to define limits of nets of subsets.

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Let \( \pi \) be a dynamical system on \( X \) and \( \pi_i \) be a net of dynamical systems on \( X \) such that \( \pi_i \rightarrow \pi \) in the following sense: if \( x_i \) and \( t_i \) are nets converging to \( x \) and \( t \) respectively, then \( x_i \pi_i t_i \rightarrow x \pi t \) [2, VI, 3.1-3.11]. If \( X \) is locally compact, then the convergence of \( \pi_i \) to \( \pi \) as defined above is equivalent to the convergence of \( \pi_i \) to \( \pi \) in the compact open topology [2, VI, 3.3]. A section with respect to \( \pi \) may not be a section with respect to any of the \( \pi_i \) [2, VI, 3.10.1].

Convention. Any set subscripted by an \( i \) is to be considered relative to \( \pi_i \); e.g., \( L_\pi^+(x) \) is the positive limit set of \( x \) with respect to \( \pi_i \).

The purpose of this paper is to prove the following theorem.

**Theorem 1.** Let \( X \) be a locally compact metric space on which there is defined a dynamical system \( \pi \) and a net \( \pi_i \) of dynamical systems such that \( \pi_i \rightarrow \pi \). If \( M \) is a compact asymptotically stable set of \( \pi \), then there are asymptotically stable sets \( M_i \) of \( \pi_i \) such that \( \limsup M_i \subseteq M \). Moreover, if \( M \) is invariant with respect to each \( \pi_i \), then \( \lim M_i = M \).

Since \( M \) is asymptotically stable there is a continuous Liapunov function \( v: A^+(M) \rightarrow \mathbb{R}_+ \) for \( M \) such that (i) \( v(x \pi t) < v(x) \) whenever \( x \notin M \) and \( t > 0 \) and (ii) \( v(x) = 0 \) whenever \( x \in M \) [3, Theorem 10]. \( \mathcal{F} \) will denote the family \( \{v^{-1}([0, r]): r > 0\} \), which is a fundamental system of neighborhoods of \( M \). It is easily verified that \( v^{-1}(r), r > 0 \), is a section with respect to \( \pi \).

The proof of the theorem depends on the following two lemmas.

**Lemma 2.** Let \( U, V \in \mathcal{F} \) be such that both are compact and \( V \subseteq \text{int} U \). Then eventually \( (\text{cl}(X-U)) \pi_i, R^- \subseteq X-V \).

**Proof.** Set \( A = X-U \) and \( B = X-V \). Evidently both are open and \( A \subseteq B \). Let \( \alpha < 0 \). By the construction of \( U \), we have \( \partial A \pi \alpha \subseteq A \) and \( A \pi_i [2 \alpha, 0] \subseteq A \subseteq B \). Since \( \pi_i \rightarrow \pi \), eventually, say \( i > i_0 \), \( \partial A \pi_i, \alpha \subseteq A \) and \( A \pi_i [2 \alpha, 0] \subseteq B \). We now show that \( A \pi_i, R^- \subseteq B \) for \( i > i_0 \). Assume not. Then there is an \( x \in \partial A \) and \( t \in \mathbb{R}^- \) such that \( x \pi_i t \in \partial B \). Set \( s = \inf \{ \tau: x \pi_i \tau \in \partial A, t < s \leq 0 \} \). Then \( t < s \) and \( x \pi_i s \in \partial A \) since \( \partial A \) is compact. Moreover, \( x \pi_i (s, t) \cap A = \emptyset \). Since \( x \pi_i s \in \partial A \) and \( x \pi_i t \in \partial B \), we have that \( t-s < 2 \alpha \) (recall \( A \pi_i [2 \alpha, 0] \subseteq B \) for \( i > i_0 \)). But \( \partial A \pi_i, \alpha \subseteq \text{int} A \). This contradicts \( \emptyset = x \pi_i (s, t) \cap A = ((x \pi_i s) \pi(t-s, 0)) \cap A \). This contradiction implies \( A \pi_i, R^- \subseteq B \) for \( i > i_0 \).
Lemma 3. Let $U \in \mathcal{F}$ be compact. Then $\text{cl}(X - U)$ is eventually a negative attractor with respect to $\pi_i$.

Proof. Let $V$, $W \in \mathcal{F}$ be compact and such that $V \subseteq \text{int} U \subseteq U \subseteq \text{int} W$ and set $A = X - W$, $B = X - U$, $C = X - V$. Each is open and $\bar{A} \subseteq B \subseteq \bar{B} \subseteq C$. Since $W$ and $V$ are in $\mathcal{F}$, for each $x \in \text{cl}(C - A)$ there is a $t(x) \in \mathbb{R}^+$ such that $x \pi t(x) \subseteq A$. We will first show that there is a $T \in \mathbb{R}^+$ such that $x \pi t(x) \cap A \neq \emptyset$ for each $x \in \text{cl}(C - A)$. Assume there is no such $T$. Then there are nets $x_t \in \text{cl}(C - A)$ and $s(x_t) \in [T, 0]$ such that $x_t \in [s(x_t), 0] \cap A = \emptyset$. Since $\text{cl}(C - A)$ is compact, we may assume that $x_t \to x \in \text{cl}(C - A)$ and, since $A$ is open and $\pi$ continuous, $x \pi t(x) \subseteq A$ eventually. This contradiction implies the existence of a $T \in \mathbb{R}^+$ such that $x \pi t(x) \cap A \neq \emptyset$ for each $x \in \text{cl}(C - A)$. Since $\text{cl}(C - A)$ is compact and $\pi^t \pi$, eventually, say $i > i_0$, $x \pi t(x) \cap A \neq \emptyset$ for each $x \in \text{cl}(C - A)$. $C$ is a neighborhood of $\beta$. $\pi \pi - \beta$ eventually (Lemma 2) so that $L_i(C) \subseteq B$. If $x \in \text{cl}(C - A)$, then for $i > i_0$, $x \pi t(x) \cap A \neq \emptyset$ and (by Lemma 2) $L_i(C) \subseteq B$ eventually. Thus eventually $L_i(C) \subseteq B$ and $B$ is eventually a negative weak attractor with respect to $\pi_i$.

Proof of Theorem 1. Let the notation be as in Lemma 3. $B$ is a negative attractor and $A_i^{-1}(B) = B \subseteq U$. Therefore $\partial A_i^{-1}(B) \subseteq U$ and $\partial A_i^{-1}(B)$ is compact. Hence $X - A_i^{-1}(B)$ is asymptotically stable with respect to $\pi_i$ [4, Theorem 3.10]. Thus we have shown that, for each $r$, $M(r) = X - v^{-1}([0, r])$ is eventually a negative attractor, $X - A_i^{-1}(M(r))$ is asymptotically stable and $X - A_i^{-1}(M(r)) \subseteq v^{-1}([0, r])$. Set $r_i = \inf\{r: M(r)$ is a negative attractor with respect to $\pi_i\}$ and $0 \leq \epsilon_i \leq r_i$ be such that $M(r_i + \epsilon_i)$ is a negative attractor with respect to $\pi_i$. Finally set $M_i = X - A_i^{-1}(M(r_i + \epsilon_i))$. $M_i \subseteq v^{-1}([0, r_i + \epsilon_i])$ and is asymptotically stable. Lemma 3 implies $r_i \to 0$. Hence $M_i \subseteq v^{-1}([0, r_i + \epsilon_i]) \to v^{-1}(0) = M$, so that $\text{lim sup} M_i \subseteq M$. If $M$ is invariant with respect to each $\pi_i$, then $L_i^+(M) \subseteq M$ so that $M \subseteq X - A_i^{-1}(M(r_i + \epsilon_i)) = M_i$. It easily follows that $M_i \to M$. This completes the proof.

Remark. It should be noted that the converse of Theorem 1 is false. That is, if $M_i$ are compact asymptotically stable sets of $\pi_i$ and if $M_i$ converges to a compact set $M$, then it does not necessarily follow that $M$ is asymptotically stable with respect to $\pi$. Let $\pi$ be a planar dynamical system with the origin as a center-focus and $C_n$ ($n = 1, 2, \cdots$) a sequence of external limit cycles which converge to the origin. $\text{cl}(\text{int} C_n)$ is asymptotically stable and $\lim \text{cl}(\text{int} C_n)$ is the origin. Finally for each positive integer $n$, set $\pi_n = \pi$. $\text{cl}(\text{int} C_n)$ is asymptotically stable with respect to $\pi_n$, but the origin is not asymptotically stable with respect to $\pi$.

We now assume that $X$ is the plane $\mathbb{R}^2$ and investigate the occurrence of critical points in the $\pi_i$. We will prove the following theorem.
Theorem 4. Let $x$ be a stable isolated critical point of a planar dynamical system $\pi$. Then there are critical points $x_i$ of $\pi_i$ such that $x_i \to x$.

The proof depends on the following three lemmas.

Lemma 5. Each stable isolated critical point possesses arbitrarily small neighborhoods bounded by either a cycle or a section with respect to $\pi$ which is a simple closed curve.

Proof. The proof follows immediately from [2, VIII, 4.1] and [2, VIII, 4.3].

Lemma 6. Let $x \in X$ possess a fundamental system $\mathcal{F}$ of neighborhoods whose boundaries are simple closed curves which are sections with respect to $\pi$. If $W$ is any neighborhood of $x$, then there is a neighborhood $V \subset W$ of $x$ such that eventually $L^+_\sigma(V) \subset W$ or $L^-_{\sigma'}(V) \subset W$.

Proof. Let $U \subset \mathcal{F}$. For $\epsilon > 0$, $\bar{\partial} \pi(\epsilon, +\infty) \subset \text{int } U$ or $\bar{\partial} \pi(-\infty, -\epsilon) \subset U$ [2, VII, 4.8]. Hence $\mathcal{F}$ contains a fundamental system $\mathcal{G}$ of compact neighborhoods of $x$ which consists entirely of positively invariant sets or of negatively invariant sets. For definiteness we will assume $\mathcal{G}$ consists of positively invariant sets. Let $V, U \in \mathcal{G}$ be such that $V \subset \text{int } U \subset \bar{\partial} \pi \subset \text{int } W$. In a manner similar to that used in the proof of Lemma 2, it can be shown that eventually $V \cap R^+ \subset U$. Thus $L^+(V) \subset \bar{\partial} \pi \subset \text{int } W$. If $\mathcal{G}$ consists of negatively invariant sets, the proof is analogous.

Lemma 7. Let $x \in X$ possess a fundamental system $\mathcal{F}$ of neighborhoods whose boundaries are cycles of $\pi$. If $W$ is any simply connected neighborhood of $x$, then there are sets $V_i \subset W$ such that eventually either $L^+_\sigma(V_i) \subset W$ or $L^-_{\sigma'}(V_i) \subset W$.

Proof. Let $U \subset \mathcal{F}$ be such that $\bar{\partial} \pi \subset \text{int } W$. Since $\bar{\partial} U$ contains no critical points with respect to $\pi$, eventually $\bar{\partial} U$ contains no critical points with respect to $\pi_i$ [2, VI, 3.7]. Let $x_0 \in \bar{\partial} U$, $T$ be the fundamental period of $x_0$ with respect to $\pi$, and $S_i$ be transversals (local sections which are arcs) with respect to $\pi_i$, and which generate neighborhoods of $x$ ([2, VI, 2.12] and [2, VII, 1.6]). Let $0 < T < \frac{1}{2} T$. Eventually $x_0 \pi_i(0, \frac{1}{2} T) \cap S_i = \emptyset$, $x_0 \pi_i(\frac{1}{2} T, \frac{3}{2} T) \cap S_i = \emptyset$, and $x \pi_i(0, \frac{3}{2} T) \subset \text{int } W$. Set $t_i = \inf \{ \tau : x \pi \pi \in S_i, \tau > \frac{1}{2} T \}$. Let $C_i$ be the simple closed curve composed of $x \pi_i(0, t_i]$ and the subarc of $S_i$ connecting $x$ and $x \pi_i(t_i)$. Clearly this can eventually be done. Finally set $V_i = \text{int } C_i$. Then $\bar{V}_i \subset W$ since $W$ is simply connected. $V_i$ is either positively invariant or negatively invariant [2, VII, 4.8]. Hence $L^+_\sigma(V_i) \subset \bar{V}_i \subset W$ or $L^-_{\sigma'}(V_i) \subset \bar{V}_i \subset W$. This completes the proof.

Proof of Theorem 4. Let $x$ be a stable isolated critical point and $W$ be a compact simply connected neighborhood of $x$. By Lemmas 5, 6 and 7 there is a net $y_i$ in $W$ such that eventually either $L^+_\sigma(y_i) \subset W$ or $L^-_{\sigma'}(y_i) \subset W$. 

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By the Poincaré-Bendixson Theorem [2, VIII, 1.14], if $L_t^+(y_i)\subset W$, then $L_t^+(y_i)$ is a cycle of $\pi_i$ or $L_t^+(y_i)$ contains critical points of $\pi_i$. A similar result holds if $L_t^-(y_i)\subset W$. If $L_t^+(y_i)$ (or $L_t^-(y_i)$) is a cycle then int $L_t^+(y_i)\subset W$ and int $L_t^+(y_i)$ contains a critical point [2, VII, 4.8]. Thus eventually $W$ contains critical points of $\pi_i$. Let $x_i$ be a critical point of $\pi_i$ and assume $x_i\to x_0$. Then, for any $t \in \mathbb{R}$,

$$x_0 \leftarrow x_i = x_i^\pi t \to x_0^\pi t.$$ 

Hence $x_0$ is a critical point of $\pi$. Since $x$ is an isolated critical point of $\pi$, the desired result easily follows.

**Remarks.** (1) If $x$ is a stable critical point of $\pi$ and $x_i$ are critical points of $\pi_i$ such that $x_i\to x$, it may be that none of the $x_i$ are stable. Let $\pi_n$ ($n=1, 2, \cdots$) be the planar dynamical system indicated by the following drawing (where the cycle is a circle of radius $1/n$). Then the $\pi_n$ can be chosen so that they converge to a dynamical system $\pi_n$ in which the origin is asymptotically stable.

(2) If $x$ is a critical point of $\pi$, it is possible that there are no critical points of the $\pi_i$ close to $x$. Let $\pi_n$ ($n=1, 2, \cdots$) be the planar dynamical system given by $\dot{x}=x^2/(1+x^2)+1/n$, $y=0$ and $\pi$ be the planar system given by $\dot{x}=x^2/(1+x^2)$, $\dot{y}=0$. Then $\pi_n\to \pi$, each $\pi_n$ is free from critical points, and $\pi$ has a critical point at the origin.

**References**