ON AN INTEGRAL FORMULA FOR CLOSED HYPERSURFACES OF THE SPHERE

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Abstract. In a compact oriented hypersurface $M^n$ of the sphere $S^{n+1}$ the integral formula $\int_{M^n} \nabla K_r \, dV = n \int_{M^n} (K_r - K_{r+1}) e \, dV$ is proved where $K_r$ is the $r$th mean curvature, $e$ is the unit normal of $M^n$ in $S^{n+1}$. Some applications are considered.

1. Let $S^{n+1}$ be the unit sphere in a euclidean space $E^{n+2}$ and $x: M^n \to S^{n+1}$ be an isometric immersion of a compact oriented Riemannian manifold $M^n$ of dimension $n$ into $S^{n+1}$. Let $F(M^n)$, $F(S^{n+1})$ and $F(E^{n+2})$ be the bundles of orthonormal frames of $M^n$, $S^{n+1}$ and $E^{n+2}$ respectively. Let $B$ be the set of elements $b = (p, e_1, \cdots, e_n, e, x(p))$ such that $(p, e_1, \cdots, e_n) \in F(M^n)$, $(x(p), e_1, \cdots, e_n, e) \in F(S^{n+1})$ and $(x(p), e_1, \cdots, e_n, e, x) \in F(E^{n+2})$ with coherent orientations. $dx(e_i)$ is identified with $e_i$, $i = 1, 2, \cdots, n$. Define $\tilde{x}: B \to F(E^{n+2})$ by $\tilde{x}(b) = (x(p), e_1, \cdots, e_n, e, x)$.

By the structure equations of $E^{n+2}$ and the pullback by $\tilde{x}$ we may write

\[
(1.1) \quad dx = \sum \omega_i e_i, \quad \quad de = \sum_i \theta_i e_i
\]

with $\theta_i = k_i w_i$. Where $i = 1, 2, \cdots, n$; $k_1, \cdots, k_n$ are principal curvatures of $M^n$ in $S^{n+1}$ at $p$. $de$ does not have component in the $x$ direction is easily followed from $d(e \cdot x) = 0$.

2. Let $\cdot \cdot \cdot \cdot \cdot$ denote the combined operation of the vector product and exterior product ([1], [3], [4]). Put

\[
(2.1) \quad \Delta_r = [de, \cdots, de, dx, \cdots, dx, e, x].
\]

Then

\[
(-1)^{n-r-1} \Delta_r = [de, \cdots, de, dx, \cdots, dx, x]^{(r+1 \text{ times})}^{(n-r-1 \text{ times})}
- [de, \cdots, de, dx, \cdots, dx, e]^{(r \text{ times})}^{(n-r \text{ times})}
\]
Using (1.1) and straight computation we have

\[ |\text{de, \cdots, de, dx, \cdots, dx, x}| = -n! K_{r+1}e \, dV \]

and

\[ |\text{de, \cdots, de, dx, \cdots, dx, e}| = n! K_r x \, dV \]

where \( dV = \omega_1 \wedge \cdots \wedge \omega_n \) is the volume element in \( M^n \) and \( K_r \) is the \( r \)th mean curvature of \( M^n \) in \( S^{n+1} \) defined by the elementary symmetric functions of \( k_1, \cdots, k_n \) as follows:

\[ \binom{n}{r} K_r = \sum_{j_1 < \cdots < j_r} k_{j_1} k_{j_2} \cdots k_{j_r} \quad (1 \leq r \leq n). \]

Thus

\[ d\Delta_r = (-1)^n n! (K_{r+1}e + K_rx) \, dV, \]

and by Stokes' theorem we have

\[ \int_{M^n} (K_{r+1}e + K_rx) \, dV = 0, \quad r = 1, 2, \cdots, n-1. \]

This integral formula (2.3) has been obtained by Reilly [5].

3. Substituting (1.1) in the right side of (2.1) yields [1]

\[ \Delta_r = (-1)^{n+1} r! (n - r - 1)! \sum_{i=0}^{r} (-1)^i \binom{n}{r - i} K_{r-i}^* U_i \]

where * is the Hodge star operation and

\[ U_i = \det \sum_j (k_j) \omega_j e_j, \]

\[ *U_i = \sum_j (-1)^{j-1} (k_j)^i \omega_1 \wedge \cdots \wedge \omega_j \wedge \cdots \wedge \omega_n e_j, \]

\( i=0, 1, \cdots, n \). Using (3.1) to calculate \( d\Delta_r \), we have

\[ d\Delta_r = (-1)^{n+1} r! (n - r - 1)! \left[ \binom{n}{r} dK_r \wedge *dx + \binom{n}{r} K_r d(*dx) \right. \]

\[ + \sum_{i=1}^{r} (-1)^i \binom{n}{r - i} d(K_{r-i}^* U_i) \].

It is easy to show that

\[ dK_r \wedge *dx = \nabla K_r \, dV, \]

\[ d(*dx) = -n(K_1 e + x) \, dV. \]
Hence we have
\[ d\Delta_r = (-1)^{n+1} r! (n - r - 1)! \]
\[ \cdot \left[ \binom{n}{r} \nabla K_r dV - n \binom{n}{r} K_1 e dV \right. \]
\[ \left. - n \binom{n}{r} K_r x dV + \sum_{i=1}^{r} (-1)^i \left( \binom{n}{r-i} d(K_{r-i} U_i) \right) \right]. \]

On the other hand by (3.1) and (2.2) we obtain
\[ dx \cdot \Delta_r = (-1)^{n+1} r! (n - r - 1)! \sum_{i=0}^{r} \left[ (-1)^i \left( \binom{n}{r-i} K_{r-i} \sum_{j} (k_j)^i \right) \right] dV, \]
\[ x \cdot d\Delta_r = (-1)^n r! K_r dV. \]
From \( x \cdot \Delta_r = 0 \) it follows \( 0 = dx(x \cdot \Delta_r) = dx \cdot \Delta_r + x \cdot d\Delta_r \) and hence
\[ r \binom{n}{r} K_r - \sum_{i=1}^{r} (-1)^i \left( \binom{n}{r-i} K_{r-i} \sum_{j} (k_j)^i \right) = 0. \]
This is the well-known identity of Newton.

Since \( d(K_e )dV = \nabla K_r dV - nK_r (K_1 e + x) dV \), one obtains by the Stokes' theorem
\[ \int_{M^n} \left[ \nabla K_r - nK_r (K_1 e + x) \right] dV = 0. \]
Together with (2.3) we have the following theorem.

**Theorem.** Let \( M^n \) be a compact oriented hypersurface in \( S^{n+1} \), \( K_r \) be the \( r \)th mean curvature of \( M^n \) in \( S^{n+1} \), \( e \) be the unit normal of \( M^n \) in \( S^{n+1} \). Then
\[ \int_{M^n} \nabla K_r dV = n \int_{M^n} (K_r K_1 - K_{r+1}) e dV. \]

4. The following are some applications of the theorem.

**Corollary 1.** In the theorem suppose, furthermore, that there is a fixed vector \( a \) in \( E^{n+2} \) such that the function \( a \cdot e \) is of the same sign on \( M^n \), \( K_i > 0 \) for \( i = 1, \cdots, r \), \( 1 \leq r \leq n-1 \), and \( K_r \) is constant. Then \( M^n \) is a hypersphere in \( S^{n+1} \).

**Proof.** Under the assumption we have that \( K_r K_1 - K_{r+1} = 0 \). The same argument as in \([4, p. 731]\) yields \( k_1 = k_2 = \cdots = k_n \) at all points of \( M^n \). Hence \( M^n \) is a hypersphere in \( S^{n+1} \).
Corollary 2. In the theorem suppose, furthermore, that $M^n$ is minimal in $S^{n+1}$ and that there is a fixed vector $a$ in $E^{n+2}$ such that the function $a \cdot e$ is of the same sign on $M^n$. Then $M^n$ is totally geodesic.

Proof. By the assumptions and (3.4) it implies that $K_1 = K_2 = 0$. So $k_i = 0$ ($i = 1, 2, \cdots, n$) and $M^n$ is totally geodesic. This result is known [2, p. 33].

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References

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