THE TOPOLOGICAL COMPLEMENTATION THEOREM À LA ZORN

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Abstract. Steiner’s topological complementation theorem is given a short simple proof using Zorn’s Lemma.

A. K. Steiner [3, Theorem 7.8, p. 397] proved that the lattice of topologies on a fixed set $X$, denoted $\Sigma(X)$, is complemented. In fact, she showed that each $t \in \Sigma$ has a complement in $\Pi(X)$, the sublattice of principal topologies. (A topology $t \in \Sigma$ is principal iff each point $x \in X$ has a smallest $t$-neighborhood: $B_t(x)$.) Her proof was quite complicated and although van Rooij [1] gave a simpler proof, his proof used both Zorn’s Lemma and two applications of transfinite induction. The purpose of this note is to prove Steiner’s Theorem via a standard Zornification by the simple trick of suitably adjoining a new point $p$ to $X$ and subsequently discarding it.

Theorem (A. K. Steiner). Every $t \in \Sigma$ has a complement $t' \in \Pi$.

Proof. Take a point $p \notin X$ and let $T$ be the topology defined on $Y = X \cup \{p\}$ by $T = t \cup \{U \cup \{p\} | U \in t\}$. Let $A = \{(A, s) | p \in A \subset Y \text{ and } s \in \Pi(A) \text{ is a complement for } T|A\}$. Then $A \neq \emptyset$, since $((\{p\}, \emptyset, \{p\}) \in A$. Partially order $A$ by $(A_1, s_1) \leq (A_2, s_2)$ iff (i) $A_1 \subset A_2$, (ii) $B_1(x) = B_2(x)$ for $x \in A_1 \setminus \{p\}$, and (iii) $B_1(p) \subset B_2(p) \subset B_1(p) \cup A_2 \setminus A_1$. If

$$B = \{(A_i, s_i) | i \in I\} \subset A$$

is totally ordered, let $(A, s)$ be defined by $A = \bigcup A_i$; $B(x) = B_1(x)$ if $x \in A_1 \setminus \{p\}$ and $B(p) = \bigcup B_i(p)$. It is easily verified that $(A, s)$ is an upper bound for $B$ in $A$, so by Zorn’s Lemma $A$ has a maximal element, say $(M, m)$. But, $M = Y$. For otherwise, if $q \in Y \setminus M$ we can extend $m$ to $m'$ on

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1 For $t_1, t_2 \in \Sigma$ we have $t_1 \leq t_2$ iff $t_1 \subset t_2$. We say that $t' \in \Sigma$ is a complement for $t \in \Sigma$ iff $t \land t' = 1$, the discrete topology, and $t \lor t' = 0$, the trivial topology. See [2] for the cardinality of the set of complements.
\[ M' = M \cup \{q\} \text{ by: } B_m(p) = B_m(q) \text{ if } M \text{ is not open in } M', B_m(p) = B_m(p) \cup \{q\} \text{ if } M \text{ is open in } M', B_m(q) = \{q\} \text{ if } \{q\} \text{ is not open in } M', B_m(q) = B_m(p) \cup \{q\} \text{ if } \{q\} \text{ is open in } M'. \]

Since \((M, m) \leq (M', m') \in \mathcal{A}\), this is a contradiction. It immediately follows that, since \(m\) is a principal complement for \(T\) and since both \(X\) and \(\{p\}\) are \(T\)-open, \(m|X\) is a principal complement for \(t\). □

References

