THE TOPOLOGICAL COMPLEMENTATION THEOREM
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Abstract. Steiner's topological complementation theorem is given a short simple proof using Zorn's Lemma.

A. K. Steiner [3, Theorem 7.8, p. 397] proved that the lattice of topologies on a fixed set $X$, denoted $\Sigma$ or $\Sigma(X)$, is complemented.1 In fact, she showed that each $t \in \Sigma$ has a complement in $\Pi = \Pi(X)$, the sublattice of principal topologies. (A topology $t \in \Sigma$ is principal iff each point $x \in X$ has a smallest $t$-neighborhood: $B_t(x)$.) Her proof was quite complicated and although van Rooij [1] gave a simpler proof, his proof used both Zorn's Lemma and two applications of transfinite induction. The purpose of this note is to prove Steiner's Theorem via a standard Zornification by the simple trick of suitably adjoining a new point $p$ to $X$ and subsequently discarding it.

Theorem (A. K. Steiner). Every $t \in \Sigma$ has a complement $t' \in \Pi$.

Proof. Take a point $p \notin X$ and let $T$ be the topology defined on $Y = X \cup \{p\}$ by $T = t \cup \{U \cup \{p\} | U \in t\}$. Let $\mathcal{A} = \{(A, s) | p \in A \subseteq Y$ and $s \in \Pi(A)$ is a complement for $T|A\}$. Then $\mathcal{A} \neq \emptyset$, since $\{(p), \emptyset, \{p\}\} \in \mathcal{A}$. Partially order $\mathcal{A}$ by $(A_1, s_1) \leq (A_2, s_2)$ iff (i) $A_1 \subseteq A_2$, (ii) $B_1(x) = B_2(x)$ for $x \in A_1 \setminus \{p\}$, and (iii) $B_1(p) \subseteq B_2(p) \subseteq B_1(p) \cup A_2 \setminus A_1$. If $\mathcal{B} = \{(A_i, s_i) | i \in I\} \subseteq \mathcal{A}$ is totally ordered, let $(A, s)$ be defined by $A = \bigcup A_i$; $B_s(x) = B_i(x)$ if $x \in A_i \setminus \{p\}$ and $B_s(p) = \bigcup B_i(p)$. It is easily verified that $(A, s)$ is an upper bound for $\mathcal{B}$ in $\mathcal{A}$, so by Zorn's Lemma $\mathcal{A}$ has a maximal element, say $(M, m)$. But, $M = Y$. For otherwise, if $q \in Y \setminus M$ we can extend $m$ to $m'$ on $M = Y$. For otherwise, if $q \in Y \setminus M$ we can extend $m$ to $m'$ on

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1 For $t_1, t_2 \in \Sigma$ we have $t_1 \leq t_2$ iff $t_1 \subseteq t_2$. We say that $t' \in \Sigma$ is a complement for $t \in \Sigma$ iff $t \wedge t' = 1$, the discrete topology, and $t \vee t' = 0$, the trivial topology. See [2] for the cardinality of the set of complements.
$M' = M \cup \{q\}$ by: $B_m(q) = B_m(p)$ if $M$ is not open in $M'$, $B_m(q) = B_m(p) \cup \{q\}$ if $M$ is open in $M'$, $B_m(q) = \{q\}$ if $\{q\}$ is not open in $M'$, $B_m(q) = B_m(p) \cup \{q\}$ if $\{q\}$ is open in $M'$. Since $(M, m) \leq (M', m') \in \mathcal{A}$, this is a contradiction. It immediately follows that, since $m$ is a principal complement for $T$ and since both $X$ and $\{p\}$ are $T$-open, $m|X$ is a principal complement for $t$. 

REFERENCES

