SIMPLY-CONNECTED BRANCHED COVERINGS OF \( S^3 \)

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Abstract. It is shown that a simply connected covering of \( S^3 \) branched over a torus knot or one of a certain class of links is \( S^3 \).

A possible approach to the 3-dimensional Poincaré conjecture is outlined by Fox in [5]: Construct a compact simply-connected covering of \( S^3 \) branched over a tame link, and then try to decide whether or not it is in fact \( S^3 \). It is the purpose of the present note to point out that if the complement of the link in question is a Seifert fibre space, then this construction will never yield a counterexample to the Poincaré conjecture. In particular, the simply-connected branched covering over the trefoil described in detail in [4] and [5] is indeed \( S^3 \). (This last fact also follows from a theorem of Burde [1].)

Lemma. Let \( M \) be a Seifert fibre space (with or without boundary), and suppose that \( K_1, \ldots, K_n \subset \text{int } M \) are fibres (ordinary or exceptional). Then if \( \tilde{M} \) is any compact covering of \( M \) branched over \( K_1 \cup \cdots \cup K_n \), \( \tilde{M} \) is also a Seifert fibre space.

Proof. Removing the interior of a tubular neighbourhood \( T_i \) of each \( K_i \), gives a Seifert fibre space \( \tilde{N} \). If \( N \) is that part of \( \tilde{M} \) which lies over \( N \), then \( \tilde{N} \) is a Seifert fibre space, and the covering projection \( p: \tilde{N} \rightarrow N \) takes fibres to fibres [6, p. 195]. The branched covering \( \tilde{M} \) is then obtained by attaching solid tori \( \tilde{T}_{i,j} \), \( j = 1, \ldots, k_i \), \( i = 1, \ldots, n \), to \( \tilde{N} \), in such a way that a meridian on \( \partial \tilde{T}_{i,j} \subset \partial \tilde{N} \) projects, under \( p \), to a meridian on \( \partial T_{i,j} \subset \partial N \) [6, pp. 231–233]. Since the fibering of \( N \) extends to a fibering of \( M \), the fibres on \( \partial T_i \) are not meridians. Therefore the fibres on \( \partial \tilde{T}_{i,j} \) are not meridians, and the fibering of \( \tilde{N} \) can be extended to a fibering of \( \tilde{M} \).

Theorem. If \( \tilde{M} \) is a compact simply-connected covering of \( S^3 \) branched over any link whose complement is a Seifert fibre space, then \( \tilde{M} \cong S^3 \).

Remark. A complete description of such links is given in [2]. In particular, the knots with this property are precisely the torus knots (see also [3]).

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287
PROOF OF THEOREM. If the fibering of the complement of the link extends to a fibering of $S^3$ in which the components of the link are fibres, then $\tilde{M}$ is a Seifert fibre space. Since it is simply-connected, it is therefore $S^3$ [6, p. 206].

If the fibering of the complement does not so extend, it can at least be extended to all components of the link except one, say $K$ [2]. If $T$ is a tubular neighbourhood of $K$, and $Q$ the complement of the interior of $T$, then $Q$ is a Seifert fibre space such that the fibres on $\partial T = \partial Q$ are meridians. Then $M = (\tilde{T}_1 \cup \cdots \cup \tilde{T}_k) \cup \tilde{Q}$, where $\tilde{T}_1, \cdots, \tilde{T}_k$ are solid tori, and $\tilde{Q}$ is a Seifert fibre space such that the fibres on $\partial \tilde{T}_j = \partial \tilde{Q}$, $j = 1, \cdots, k$, are meridians. Now $\pi_1(M) \cong \pi_1(\tilde{Q})/\langle h \rangle$, where $h$ is the element represented by an ordinary fibre of $\tilde{Q}$, and by hypothesis this is the trivial group. The argument in [2, p. 90] then shows that in fact $k = 1$, the Zerlegungsfläche is a disc, and there are no exceptional fibres. Hence $\tilde{Q} \cong D^2 \times S^1$, and $M \cong (S^1 \times D^2) \cup (D^2 \times S^1)$, identified along $S^1 \times S^1$ by the identity, which is just $S^3$.

REFERENCES


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