SIMPLY-CONNECTED BRANCHED COVERINGS OF $S^3$

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Abstract. It is shown that a simply connected covering of $S^3$ branched over a torus knot or one of a certain class of links is $S^3$.

A possible approach to the 3-dimensional Poincaré conjecture is outlined by Fox in [5]: Construct a compact simply-connected covering of $S^3$ branched over a tame link, and then try to decide whether or not it is in fact $S^3$. It is the purpose of the present note to point out that if the complement of the link in question is a Seifert fibre space, then this construction will never yield a counterexample to the Poincaré conjecture. In particular, the simply-connected branched covering over the trefoil described in detail in [4] and [5] is indeed $S^3$. (This last fact also follows from a theorem of Burde [1].)

Lemma. Let $M$ be a Seifert fibre space (with or without boundary), and suppose that $K_1, \cdots, K_n \subseteq \text{int } M$ are fibres (ordinary or exceptional). Then if $\bar{M}$ is any compact covering of $M$ branched over $K_1 \cup \cdots \cup K_n$, $\bar{M}$ is also a Seifert fibre space.

Proof. Removing the interior of a tubular neighbourhood $T_i$ of each $K_i$ gives a Seifert fibre space $N$. If $\bar{N}$ is that part of $\bar{M}$ which lies over $N$, then $\bar{N}$ is a Seifert fibre space, and the covering projection $p: \bar{N} \to N$ takes fibres to fibres [6, p. 195]. The branched covering $\bar{M}$ is then obtained by attaching solid tori $\hat{T}_{i,j}$, $j=1, \cdots, k_i$, $i=1, \cdots, n$, to $\bar{N}$, in such a way that a meridian on $\partial \hat{T}_{i,j} \subseteq \partial \bar{N}$ projects, under $p$, to a meridian on $\partial T_i \subseteq \partial N$ [6, pp. 231–233]. Since the fibering of $N$ extends to a fibering of $M$, the fibres on $\partial T_i$ are not meridians. Therefore the fibres on $\partial \hat{T}_{i,j}$ are not meridians, and the fibering of $\bar{N}$ can be extended to a fibering of $\bar{M}$.

Theorem. If $\bar{M}$ is a compact simply-connected covering of $S^3$ branched over any link whose complement is a Seifert fibre space, then $\bar{M} \cong S^3$.

Remark. A complete description of such links is given in [2]. In particular, the knots with this property are precisely the torus knots (see also [3]).

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Proof of Theorem. If the fibring of the complement of the link extends to a fibring of \( S^3 \) in which the components of the link are fibres, then \( \tilde{M} \) is a Seifert fibre space. Since it is simply-connected, it is therefore \( S^3 \) [6, p. 206].

If the fibring of the complement does not so extend, it can at least be extended to all components of the link except one, say \( K \) [2]. If \( T \) is a tubular neighbourhood of \( K \), and \( Q \) the complement of the interior of \( T \), then \( Q \) is a Seifert fibre space such that the fibres on \( \partial T = \partial Q \) are meridians. Then \( \tilde{M} = (\tilde{T}_1 \cup \cdots \cup \tilde{T}_k) \cup \tilde{Q} \), where \( \tilde{T}_1, \cdots, \tilde{T}_k \) are solid tori, and \( \tilde{Q} \) is a Seifert fibre space such that the fibres on \( \partial \tilde{T}_j \subset \partial \tilde{Q} \), \( j = 1, \cdots, k \), are meridians. Now \( \pi_1(\tilde{M}) \cong \pi_1(\tilde{Q})/\langle h \rangle \), where \( h \) is the element represented by an ordinary fibre of \( \tilde{Q} \), and by hypothesis this is the trivial group. The argument in [2, p. 90] then shows that in fact \( k = 1 \), the Zerlegungsfläche is a disc, and there are no exceptional fibres. Hence \( \tilde{Q} \cong D^2 \times S^1 \), and \( \tilde{M} \cong (S^1 \times D^3) \cup (D^2 \times S^1) \), identified along \( S^1 \times S^1 \) by the identity, which is just \( S^3 \).

References


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