ERROR BOUNDS FOR GALERKIN'S METHOD FOR MONOTONE OPERATOR EQUATIONS

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Abstract. An abstract theorem, generalizing a result of Nitsche, is proved. This gives sharp error bounds for the Galerkin method for approximating the solutions of a large class of nonlinear operator equations in Hilbert spaces.

Let $H$ be a real Hilbert space and $T$ be a strongly monotone operator on $H$ in the sense of Browder, i.e.,

$$|(Tu - Tv, u - v)_H| \geq \gamma \|u - v\|^2_H$$

for all $u, v \in H$ and some constant $\gamma > 0$. We are interested in numerically approximating the solution of the problem of finding $u \in H$ such that

$$Tu = f,$$

where $f$ is given in $H$,

by the Galerkin method. Given a finite-dimensional subspace, $S$, of $H$, the Galerkin method is to find $u_S \in S$ such that

$$\langle Tu_S, y \rangle_H = \langle f, y \rangle_H, \quad \text{for all } y \in S.$$

From [1] and [2], we recall the following result.

Theorem 1. If $T$ is uniformly Lipschitz continuous for bounded arguments, i.e., given $B > 0$, there exists a positive constant, $C(B)$, depending only on $B$, such that $\|Tw - Tv\|_H \leq C(B)\|w - v\|_H$ for all $w, v \in H$ with $\|w\|_H \leq B$ and $\|v\|_H \leq B$, then problems (2) and (3) have unique solutions and

$$\|u - u_S\|_H \leq \gamma^{-1}C(\|f - T0\|_H) \inf_{y \in S} \|u - y\|_H.$$

In many applications, $H$ is a closed subspace of $W^{m,2}(\Omega)$ for some $m \geq 1$ and (4) yields an error bound in the $W^{m,2}$-norm when we are really interested in an error bound in the $L^2$-norm. While the bound in (4) does induce an error bound in the $L^2$-norm, one might expect that such a bound is not sharp and indeed that is the case. In this note, we present an

Let $V$ and $W$ be two real Hilbert spaces such that $V \subset H \subset W$ and there exists a positive constant, $K$, such that

$$\|h\|_W \leq K \|h\|_H, \quad \text{for all } h \in H. \quad (5)$$

As a concrete example, one may take $H \equiv W_0^{m,2}(\Omega), \quad V \equiv W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$, and $W \equiv L^2(\Omega)$.

Instead of (2), we consider the problem of finding $u \in H$ such that

$$(Tu, \phi)_H = (g, \phi)_W, \quad \text{for all } \phi \in H, \quad (6)$$

where $g$ is given in $W$. Because of (5), problem (6) is a special case of problem (3).

Our new result is

**Theorem 2.** Let $C$ be a collection of finite-dimensional subspaces, $S$, of $H$ such that if $u_S$ denotes the Galerkin approximation to $u$ in $S$, then there exist $0 < \gamma \leq \Lambda$ independent of $S$ in $C$ and a bilinear form $b_S$ on $H$ such that

(i) $(Tu - Tu_S, \phi)_H = b_S(u - u_S, \phi)$ for all $\phi \in H$ and all $S \in C$,

(ii) $b_S(\phi, \phi) \geq \gamma \|\phi\|_H^2$ for all $\phi \in H$,

(iii) $|b_S(w, v)| \leq \Lambda \|w\|_H \|v\|_H$ for all $w, v \in H$,

(iv) if $b_S(w, \phi_S) = (g, w)_W$ for all $w \in H$, then there exists a positive constant, $\rho$, independent of $S$ in $C$, such that $\|\phi\|_V \leq \rho \|g\|_W$ and

(v) there exists a positive function, $E$, on $S$ such that

$$\inf_{S \in C} g - y \|H \leq E(S) \|g\|_V, \quad \text{for all } S \in C \text{ and all } g \in V. \quad (7)$$

Then

$$\|u - u_S\|_H \leq \gamma^{-1} C^2(\|f - T0\|_H)E(S) \inf_{S \in C} \|u - y\|_H. \quad (8)$$

**Proof.** For each $S \in C$, let $e_S = u - u_S$ and consider the problem of finding $\phi_S \in H$ such that

$$(Tu, \phi_S) = (e_S, e_S)_W, \quad \text{for all } w \in H. \quad (8)$$

By our hypotheses on $b_S$, this problem has a unique solution, $\phi_S$, and $\|\phi_S\|_H \leq \rho$.

Setting $w = e_S$, we have $\|e_S\|_W = b_S(e_S, \phi_S) = (Tu - Tu_S, \phi_S)$. Moreover, by the definition of the Galerkin method, we have

$$\|e_S\|_H = (Tu - Tu_S, \phi_S - y)_H, \quad \text{for all } y \in S.$$
Thus,
\[ \|e_S\|_W \leq C(\|f - T0\|_{H^1}) \|u - u_S\|_{H^1} \|\phi - \psi\|_{H^1}, \quad \text{for all } y \in S. \]

Using Theorem 1 to bound \( \|u - u_S\|_{H^1} \) and (7), we obtain the required result. Q.E.D.

The reader is referred to [6] for further details and applications of this result to boundary value problems for linear and semilinear elliptic partial differential equations and eigenvalue problems.

REFERENCES