TWO OBSERVATIONS ON THE CONGRUENCE EXTENSION PROPERTY

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Abstract. A pair of algebras \( \mathfrak{A}, \mathfrak{B} \) with \( \mathfrak{B} \) a subalgebra of \( \mathfrak{A} \) is said to have the (Principal) Congruence Extension Property (abbreviated as PCEP and CEP, respectively) if every (principal) congruence relation of \( \mathfrak{B} \) can be extended to \( \mathfrak{A} \). A pair of algebras \( \mathfrak{A}, \mathfrak{B} \) is constructed having PCEP but not CEP, solving a problem of A. Day. A result of A. Day states that if \( \mathfrak{B} \) is a subalgebra of \( \mathfrak{A} \) and if for any subalgebra \( \mathfrak{C} \) of \( \mathfrak{A} \) containing \( \mathfrak{B} \), the pair \( \mathfrak{A}, \mathfrak{C} \) has PCEP, then \( \mathfrak{A}, \mathfrak{B} \) has CEP. A new proof of this theorem that avoids the use of the Axiom of Choice is also given.

1. The example. Let \( \mathfrak{A} = \{a, b, c, d, e, f\} \). We define a binary operation + on \( \mathfrak{A} \) by \( a+f = e, \; b+f = e, \; x+y = x \) otherwise. Let \( \mathfrak{A} = \langle A; + \rangle \) and \( \mathfrak{B} = \{a, b, c, d\} \). Then \( \mathfrak{B} \) is a subalgebra of \( \mathfrak{A} \). An easy computation shows that \( \mathfrak{A}, \mathfrak{B} \) has PCEP. Now let \( \Theta = \Theta_b(a, c) \lor \Theta_b(b, d) \). Then \( c \not\equiv d(\Theta) \). However, if \( \Theta \) denotes the smallest congruence of \( \mathfrak{A} \) with \( \Theta \geq \Theta_b \), then \( a \equiv c(\Theta) \); hence \( a+f \equiv c+f(\Theta) \), that is, \( e \equiv c(\Theta) \). Similarly, \( b \equiv d(\Theta) \), and so \( e \equiv d(\Theta) \). By transitivity, \( e \equiv d(\Theta) \). Thus \( \Theta \not\subseteq \Theta \). This means that the pair \( \mathfrak{A}, \mathfrak{B} \) does not have CEP.

2. The proof. We want to prove the following important

Theorem (A. Day [1]). Let \( \mathfrak{A} \) be an algebra and \( \mathfrak{B} \) a subalgebra of \( \mathfrak{A} \). If PCEP holds for any pair \( \mathfrak{A}, \mathfrak{C} \) where \( \mathfrak{C} \) contains \( \mathfrak{B} \), then \( \mathfrak{A}, \mathfrak{B} \) has CEP.

Our proof, as well as Day’s, is based on the following (A. W. Goldie [3], see also G. Grätzer [4] and Exercise 64 of Chapter 1 in G. Grätzer [5]):

Lemma. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be algebras and let \( \mathfrak{B} \) be a subalgebra of \( \mathfrak{A} \). Let \( \Phi \) be a congruence relation of \( \mathfrak{A} \) and \( \Theta \) be a congruence relation of \( \mathfrak{B} \) satisfying \( \Phi \supseteq \Theta \). Set \( D = \{B|\Phi = \{x|x \in A, x \equiv y(\Phi) \text{ for some } y \in B\} \}. \) We define
a binary relation $\Theta(\Phi)$ on $D$ by the rule $u \equiv v(\Theta(\Phi))$ iff $u \equiv x(\Phi)$, $x \equiv y(\Theta)$, $y \equiv v(\Phi)$ for some $x, y \in B$. Then $D$ is a subalgebra of $A$ and $\Theta(\Phi)$ is a congruence relation on $D$. Furthermore, $(\Theta(\Phi))_B = \Theta$.

Proof of the Theorem. Let $A$ and $B$ be given as in the Theorem. We shall prove that, for any subalgebra $C$ of $A$ containing $B$, the pair $A, C$ has CEP. Let $\Theta$ be a congruence relation on $C$ and let $\bar{\Theta}$ be the smallest congruence relation on $A$ satisfying $\bar{\Theta}_C \geq \Theta$. Obviously,

$$\bar{\Theta} = \lor (\Theta_A(x, y) \mid x, y \in C \text{ and } x \equiv y(\Theta)).$$

We want to show that

(1) for $a, b \in C$, $a \equiv b(\bar{\Theta})$ implies that $a \equiv b(\Theta)$

(this is CEP). In view of the formula for $\bar{\Theta}$ and the way joins of congruences can be described, (1) is equivalent to:

(2) For any subalgebra $C$ of $A$ with $B \subseteq C$, if $a, b \in C$, $a_1, b_1, \ldots, a_n, b_n \in C$, $a_i \equiv b_i(\Theta)$ for $i = 1, \ldots, n$, $a = x_0, x_1, \ldots, x_n = b$, $x_i \in A$ for $i = 1, \ldots, n - 1$, and $x_{i-1} \equiv x_i(\Theta_C(a_i, b_i))$ for $i = 1, \ldots, n$, then $a \equiv b(\Theta)$.

We prove this statement by induction on $n$. For $n = 1$ it is obvious since $A, C$ has PCEP. Now assume that $n > 1$ and that the statement is valid for $n - 1$. Set $D = [C]^{\Theta_A(a_n, b_n)}$, $\Theta_0 = \Theta_\emptyset(a_1, b_1) \lor \cdots \lor \Theta_\emptyset(a_n, b_n)$. Since PCEP holds for $A, C$, we have $(\Theta_A(a_n, b_n))_C \leq \Theta_0$; hence we can form $\Psi = \Theta_\emptyset(\Theta_A(a_n, b_n))$ and it will satisfy $\Psi_C = \Theta_0$. Now observe that $A, D$, $a = x_0, \ldots, x_{n-1}, a_1, b_1, \ldots, a_{n-1}, b_{n-1}$, and $\Psi$ satisfy the assumptions of (2) with $n - 1$, hence we can conclude that $a \equiv x_{n-1}(\Psi)$. Obviously, $x_{n-1} \equiv x_n(\Psi)$, hence $a \equiv b(\Psi)$. Since $a, b \in C$ and $\Psi_C = \Theta_0 \leq \Theta$ we conclude that $a \equiv b(\Theta)$, completing the proof of (2). If we now let $C = B$ the theorem follows.

References


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