TENSOR PRODUCTS OF QUATERNION ALGEBRAS*

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Abstract. Two quaternion division algebras have a common quadratic subfield if their tensor product contains zero-divisors.

Let $Q_1$ and $Q_2$ be quaternion division algebras over a field $K$. If $Q_1$, $Q_2$ have a common quadratic subfield, it is evident that $Q_1 \otimes Q_2$ contains zero-divisors (all tensor products are taken over $K$). In this note we shall prove that the converse is true.

Theorem. Let $Q_1$, $Q_2$ be quaternion division algebras with center $K$. Suppose that $Q_1 \otimes Q_2$ is not a division algebra. Then $Q_1$ and $Q_2$ possess a common quadratic subfield.

Proof. Let $K(u)$ be a separable quadratic subfield of $Q_2$, and let the nontrivial automorphism of $K(u)$ over $K$ be given by $u \mapsto u'$. We complete a generation of $Q_2$ with an element $v$ such that $uv = vu$. We have that $uv = a$ and $v^2 = b$, where $a$ and $b$ are nonzero elements of $K$. Write $L = Q_1 \otimes K(u)$. If $L$ is not a division algebra, then $Q_1$ contains a subfield isomorphic to $K(u)$, and we are done. So the proof need only continue on the assumption that $L$ is a division algebra.

We have that $Q_1 \otimes Q_2$ is the vector space direct sum of $L$ and $Lv$. The given zero-divisor thus has the form $p + qv$ with $p, q \in L$. Necessarily $q \neq 0$, and we can renormalize to make $q = 1$. Write $p = c + du$, with $c, d \in Q_1$, and set $p^* = c + du'$. Note that $u$ and $v$ commute with $c$ and $d$. Thus we have $vp = p^* v$ (and also $pv = vp^*$). Hence $(p + v)(p^* - v) = pp^* - v^2 = pp^* - b$. This element is a zero-divisor and lies in $L$; hence it is 0. So

\[
b = pp^* = (c + du)(c + du')
= c^2 + d^2a + dcu + cdv'
= c^2 + d^2a + (dc - cd)u + cd(u + u').
\]

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Since \( u + u' \in K \), we deduce from this that \( c \) and \( d \) commute. We next note that

\[
(pv)^2 = pvvp = pp^*v^2 = b^2.
\]

Write \( F \) for the field generated over \( K \) by \( c \) and \( d \). If \( F \otimes Q_2 \) is a division algebra, we can deduce from \((pv)^2 = b^2\) that \( pv = \pm b^2 \), \( p = \pm v \), a contradiction. Hence \( F \otimes Q_2 \) is not a division algebra, and it follows that \( Q_2 \) contains a quadratic subfield isomorphic to \( F \), as desired.

I discovered this theorem some time ago. There appears to be some continuing interest in it, and I am therefore publishing it now.

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