QUOTIENT RINGS OF ENDOMORPHISM RINGS OF
MODULES WITH ZERO SINGULAR SUBMODULE

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Abstract. Throughout this paper \((R, M, N, S)\) will denote a
Morita context satisfying a certain nonsingularity condition. For
such contexts we give necessary and sufficient conditions in terms of
\(M\) and \(R\) for \(S\) to have a semisimple maximal left quotient ring;
respectively a full linear maximal left quotient ring, a semisimple
classical left quotient ring. In doing so we extend the corresponding
well-known theorems for rings (employing them in the process) to
endomorphism rings.

Suppose \((R, M, N, S)\) is a Morita context ([1], [2]). That is suppose
\(RMS\) and \(SN_R\) are bimodules with an \(R\)-\(R\) bimodule homomorphism
\((\cdot, \cdot): M \otimes_R N \rightarrow R\) and an \(S\)-\(S\) bimodule homomorphism \([\cdot, \cdot]: N \otimes_R M \rightarrow S\)
satisfying
\[
m_1(n_1, m_2) = (m_1, n_1)m_2 \quad \text{and} \quad n_1(m_1, n_2) = [n_1, m_1]n_2
\]
for all \(m_1, m_2 \in N\) and \(n_1, n_2 \in N\).

Throughout, unless otherwise indicated, \(M\) and \(N\) will satisfy the
following condition: \(M_S\) is faithful; and \([N, m] = 0\) for \(m \in M\) implies
that \(m = 0\).

Note that when this condition is satisfied, we can (and will) assume
that \(S \subseteq \text{Hom}_R(M, M)\).

Let \(RM\) be any left \(R\)-module, and set \(N = \text{Hom}_R(M, R)\) and \(S = \text{Hom}_R(M, M)\). Set \((m, f) = (m)f\) for \(m \in M, f \in N\); and \([f, m]\) is defined
via \(m_1[f, m] = (m_1, f)m\) for all \(m_1, m_1 \in M, f \in N\). Then \((R, M, N, S)\) is a
Morita context, called the standard context for \(RM\).

If \(R\) is semiprime and \(RM\) is torsionless, then the above condition is
satisfied by the standard Morita context for \(RM\). If \(RM\) is a generator
and \(1 \in R\); or indeed, if \((\text{Trace}_RM)m \neq 0\) whenever \(0 \neq m \in M\), then the
standard Morita context for \(RM\) satisfies the above condition.

Lemma 1. (a) If \(A\) is an essential left ideal of \(S\), then \(MA\) is an essential
submodule of \(RM\).
(b) If \( K \) is an essential submodule of \( _RM \), then
\[
[N, K] = \left\{ \sum_{i=1}^{t} [n_i, k_i] \mid n_i \in N, k_i \in K \right\}
\]
is an essential left ideal of \( S \).

**Proof.** (a) If \( 0 \neq m \in M \), then \([N, m] \subseteq A^0 \). Since \( M_S \) is faithful,
\[0 \neq M([N, m] \cap A) \subseteq M[N, m] \cap MA = (M, N)m \cap MA \subseteq Rm \cap MA.
\]
(b) If \( 0 \neq s \in S \), then \( Ms \cap K \neq 0 \). Hence \( 0 \neq [N, Ms \cap K] \subseteq [N, Ms] \cap [N, K] \subseteq SS \cap [N, K] \).

**Proposition 2.** \( Z(_RM) = 0 \) if and only if \( Z(S) = 0 \).

**Proof.** Suppose \( Z(_RM) = 0 \), and let \( s \in Z(S) \). Then \( As = 0 \) where \( A \)
is an essential left ideal of \( S \). By the preceding lemma, \( MA \) is an essential
submodule of \( _RM \), and clearly \( (MA)_s = 0 \). Hence \( \ker s \) is essential in \( _RM \),
and since \( Z(_RM) = 0 \) it follows that \( s = 0 \).

Conversely suppose that \( Z(_RM) \neq 0 \). If \( 0 \neq z \in Z(_RM) \) it follows from
the condition on \([ , ]\) that there is an \( n \in N \) with \( 0 \neq [n, z] \). Set \( s = [n, z] \).
We claim that \( s \in Z(S) \), and by the previous lemma it suffices to prove
that \( \ker s \) is essential in \( _RM \). If \( m \in M \), \( m \notin \ker s \), then \( (m, s) \neq 0 \). Since
\( z \in Z(_RM) \) there exists \( a \in R^1 \) (\( R^1 \) denotes \( R \) with identity adjoined in the
customary manner) with \( a(m, n) \neq 0 \) and \( a(m, n)z = 0 \). But then \( am = an \in \ker s \) and \( s \in _RM \).

**Proposition 3** [1, p. 276]. \( d(_RM) = d(S) \).

**Proof.** Here \( d(_RM) \) denotes the (Goldie) dimension of \( _RM \). If \( \oplus_i A_i \)
is an internal direct sum of nonzero left ideals of \( S \), then a routine calculation
shows that \( \sum_i MA_i \) is a direct sum of nonzero submodules of \( _RM \). Hence \( d(S) \leq d(_RM) \).
On the other hand if \( \oplus_i M_i \) is an internal direct sum of nonzero submodules of \( _RM \) and \( A_i = [N, M_i] \), then \( \sum_i A_i \) is a
direct sum of nonzero left ideals of \( S \). Hence \( d(_RM) \leq d(S) \).

Let \( \Lambda = \text{Hom}_R(M, M) \) and \( \Omega = \text{Hom}_R(M, M) \) where \( M \) is the injective
hull of \( _RM \). As we have already noted, we can assume that \( S \subseteq \Lambda \). When
\( Z(_RM) = 0 \), \( \Omega \) is a regular self-injective ring [5] and we can assume
that \( \Lambda \subseteq \Omega \).

**Proposition 4.** When \( Z(_RM) = 0 \), \( \Omega \) is the maximal left quotient ring
of \( S \).

**Proof.** Given \( 0 \neq o \in \Omega \), \( M_o^{-1} \cap M \) is an essential submodule of \( _RM \),
and so \( (M_o^{-1} \cap M)o \neq 0 \). Hence
\[0 \neq [N, (M_o^{-1} \cap M)o] = [N, M_o^{-1} \cap M]o \subseteq S \cap S_o.
\]
Since $Z(sS) = 0$ it follows that $\Omega$ is a maximal left quotient ring of $S$.

**Theorem 5.** $S$ has a semisimple maximal left quotient ring (necessarily isomorphic to $\Omega$) if and only if $Z(RM) = 0$ and $d(RM) < \infty$.

**Proof.** By [8, Theorem 1.6], $S$ has a semisimple maximal left quotient ring if and only if $Z(sS) = 0$ and $d(sS) < \infty$. The theorem then follows from the previous three propositions.

A submodule $K$ of $RM$ is closed if $K$ has no proper essential extensions in $RM$. Let $\mathcal{C}(RM)$ denote the set of closed submodules of $RM$. If $Z(RM) = 0$ it is well known that $\mathcal{C}(RM)$ is a complete complemented lattice and $\mathcal{C}(RM)$ is lattice isomorphic to $\mathcal{C}(\Omega)$ under contraction [3, p. 61].

**Proposition 6.** For any module $RM$ (not necessarily satisfying the standing hypothesis) with $Z(RM) = 0$:

(a) If $A \in \mathcal{C}(\Omega)$, then $MA \in \mathcal{C}(\Omega)$.

(b) If $K \in \mathcal{C}(RM)$, then $\text{Hom}_R(M, K) \in \mathcal{C}(\Omega)$.

(c) $\mathcal{C}(RM)$ is lattice isomorphic to $\mathcal{C}(\Omega)$.

**Proof.**

(a) Since $\Omega$ is regular left self-injective $A = \Omega \varepsilon$ where $\varepsilon^2 = \varepsilon \in \Omega$. Then $MA = M\varepsilon$, a direct summand of $RM$ and hence closed.

(b) $K = M\varepsilon$ for some $\varepsilon^2 = \varepsilon \in \Omega$, and then $\text{Hom}_R(M, K) = \Omega \varepsilon$ which is closed.

(c) This follows from the preceding correspondence.

**Corollary 7.** If $Z(RM) = 0$, then $\mathcal{C}(RM)$ is lattice isomorphic to $\mathcal{C}(sS)$.

**Proof.** $\mathcal{C}(sS) \cong \mathcal{C}(s\Omega) \cong \mathcal{C}(\Omega)$ by [3, p. 61 and p. 70] and Proposition 4. Hence, by Proposition 6, $\mathcal{C}(RM) \cong \mathcal{C}(RM) \cong \mathcal{C}(\Omega) \cong \mathcal{C}(sS)$.

A module $RM$ is atomic if each nonzero element of $\mathcal{C}(RM)$ contains a minimal nonzero element of $\mathcal{C}(RM)$, called an atom. A ring is a full linear ring if it is isomorphic to the full ring of linear transformations of a left vector space over a division ring.

**Theorem 8.** $S$ has a maximal left quotient ring which is a direct product of full linear rings if and only if $Z(RM) = 0$ and $RM$ is atomic.

**Proof.** By [6, Theorem 2], $S$ has a maximal left quotient ring which is a direct product of full linear rings if and only if $Z(sS) = 0$ and $sS$ is atomic. By virtue of Corollary 7, $sS$ is atomic if and only if $RM$ is. The result follows from Proposition 2.

A module $RM$ is $Q$-prime if for any atoms $K_1$ and $K_2$ of $\mathcal{C}(RM)$ there exist nonzero isomorphic submodules of $K_1$ and $K_2$ respectively.

**Proposition 9.** Suppose $Z(RM) = 0$ and $RM$ is atomic. Then $RM$ is $Q$-prime if and only if all atoms of $\mathcal{C}(RM)$ are isomorphic; equivalently...
if and only if all atoms of \( \mathcal{C}(\Omega) \) are isomorphic. (For the preceding, the standing hypothesis need not hold.) Consequently, \( R_M \) is \( Q \)-prime if and only if \( S \) is \( Q \)-prime.

**Proof.** If \( R_M \) is \( Q \)-prime and \( K_1 \) and \( K_2 \) are atoms of \( R_M \), then \( K_1 \cap M \) and \( K_2 \cap M \) are atoms of \( R_M \). \( K_1 \cap M \) and \( K_2 \cap M \) contain nonzero isomorphic submodules, and so \( K_1 \cong K_2 \) since \( K_1 \) and \( K_2 \) are injective. Conversely, suppose all atoms of \( R_M \) are isomorphic. If \( L_1 \) and \( L_2 \) are atoms of \( R_M \), then there exist isomorphic atoms \( K_1 \) and \( K_2 \) of \( R_M \) such that \( K_i \cap M = L_i \), \( i = 1, 2 \). If \( f: K_1 \to K_2 \) is an isomorphism, then \( X_1 = L_1 f^{-1} \cap L_1 \) and \( X_2 = L_1 f \cap L_2 \) are nonzero isomorphic submodules of \( L_1 \) and \( L_2 \). Hence \( R_M \) is \( Q \)-prime.

Now suppose that all atoms of \( R_M \) are isomorphic and \( A_1, A_2 \) are atoms of \( \mathcal{C}(\Omega) \). By Proposition 6, \( MA_1 \) and \( MA_2 \) are in \( \mathcal{C}(R_M) \). Let \( \varphi: MA_1 \to MA_2 \) be an isomorphism. Define \( \theta \) from \( A_1 = \text{Hom}_R(R_M, MA_1) \) into \( A_2 = \text{Hom}_R(R_M, MA_2) \) by \( \psi \theta = \psi \cdot \varphi \) for \( \psi \in A_1 \). A routine calculation shows that \( \theta \) is an isomorphism from \( A_1 \) onto \( A_2 \).

Suppose all atoms of \( \Omega \) are isomorphic and let \( K_1, K_2 \) be atoms of \( \mathcal{C}(R_M) \). As in Proposition 6, \( K_i = \overline{K}_i \in \Omega \) where
\[
e_i^2 = \epsilon_i \in \Omega \quad \text{and} \quad \text{Hom}_R(\Omega, K_i) = \Omega \epsilon_i \in \mathcal{C}(\Omega)
\]
for \( i = 1, 2 \). Since \( \Omega \epsilon_1 \subseteq \Omega \epsilon_2 \) there exist \( \mu, \nu \in \Omega \) such that \( \mu \nu = \epsilon_1 \) and \( \nu \mu = \epsilon_2 \) [7, p. 63]. If \( \mu' = \mu|_{K_1} \) and \( \nu' = \nu|_{K_2} \), then \( \mu' \nu' = \text{id}_{K_1} \) and \( \nu' \mu' = \text{id}_{K_2} \). Thus \( K_1 \cong K_2 \) and all atoms of \( R_M \) are isomorphic.

**Theorem 10.** \( S \) has a maximal left quotient ring which is a full linear ring if and only if \( Z(R_M) = 0, R_M \) is atomic, and \( R_M \) is \( Q \)-prime.

**Proof.** By previous propositions \( R_M \) has the above properties if and only if \( S \) does. By [6, Theorem 1] this is the case exactly when \( S \) has a full linear ring as its maximal left quotient ring.

**Theorem 11.** \( S \) has a classical left quotient ring which is simple (semisimple) with minimum condition if and only if \( \bar{R} = R/\text{ann } R_M \) is prime (semiprime), \( m[N, M] = 0 \) for \( m \in M \) implies that \( m = 0 \), \( Z(R_M) = 0 \), and \( d(R_M) < \infty \).

**Proof.** By a well-known theorem [4], \( S \) has a classical left quotient ring which is simple (semisimple) with minimum condition if and only if \( S \) is a prime (semiprime) ring, \( Z(S) = 0 \), and \( d(S) < \infty \). In view of the earlier propositions, it suffices to prove that \( S \) is prime (semiprime) exactly when \( \bar{R} \) is prime (semiprime) and \( m[N, M] = 0 \) for \( m \in M \) implies that \( m = 0 \).

Suppose that \( S \) is prime and let \( r, r' \in R/\text{ann } R_M \). Then \( rM \neq 0, r'M \neq 0 \), and consequently \( [N, rM] \neq 0, [N, r'M] \neq 0 \). Since \( S \) is prime...
$0 \neq [N, rM][N, r'M] = [N, r(M, N)r'M]$. In particular $r(M, N)r'M \neq 0$ proving that $\bar{r}(M, N)r' \neq 0$ in $\bar{R}$ ($\bar{r}$ denotes the coset of $r$ in $R$). Hence $\bar{R}$ is prime. Next suppose that $m[N, M] = 0$ with $m \in M$. Then $[N, m][N, M] = 0$, and since $S$ is prime $[N, m] = 0$ proving that $m = 0$.

Conversely, suppose that $\bar{R}$ is prime and that $m[N, M] = 0$ for $m \in M$ implies that $m = 0$. Let $0 \neq s \in S$, $0 \neq t \in S$. Then $Ms \neq 0$ and $Mt \neq 0$ so $Ms[N, M] \neq 0$ and $Mt[N, M] \neq 0$. Thus $(Ms, N) \notin \operatorname{ann}_R M$ and $(Mt, N) \notin \operatorname{ann}_R M$. Since $R$ is prime $(Ms, N)(Mt, N) = M(s[N, M]t, N) \notin \operatorname{ann}_R M$. In particular, $s[N, M]t \neq 0$ proving that $S$ is prime.

The semiprime case is obtained by taking $r = r'$ and $s = t$ in the above proof.

We remark that [9, Theorem 2.3] is a special case of the preceding theorem.

REFERENCES


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