ON THE LOCATION OF ZEROS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. The paper considers the location of zeros of the equation \((x(t)x')' + \gamma(t)x = 0, \ t \in [t_0, t_1]\). The following theorem is proved. Let \([a, a+T], \ T = na \ (n \ a \ positive \ integer)\), be a subset of \([t_0, t_1]\). Denote \(\omega = \pi/T\). Let the coefficient functions obey the inequality
\[
\int_a^{a+T} (\gamma(t) - \omega^2 \alpha(t) \sin^2(\omega t)) \ dt > \omega^2 \int_a^{a+T} (\alpha(t) \cos(2\omega t)) \ dt.
\]
Then every solution of this equation will have a zero on \([a, a+T]\).
A more general form of this theorem is also proved.

0. Summary. This note provides a corollary to Leighton’s variational theorem, providing a sufficient condition for the existence of a zero on an interval of given length for a second-order selfadjoint equation.

1. The selfadjoint linear differential equation. We consider the equation
\[
L(x) = (\alpha(t)x')' + \gamma(t)x = 0,
\]
\(t \in [t_0, t_1], \ \alpha(t) \in C^1[t_0, t_1], \ \gamma(t) \in C[t_0, t_1] \ (\equiv d/dt), \) where the possibility \(t_1 = +\infty\) is not excluded.

We wish to find an answer to the following problem: Does every (classical) solution of (1) vanish on every interval of length \(T \ (T < (t_1 - t_0))\)? This question is not answered completely in this paper, but a sufficient condition is given for the existence of zeros on every subinterval of \([t_0, t_1]\) of length \(T\). We shall denote by \(\omega\) the number: \(\omega = \pi/T\).

Theorem 1. Let \([a, a+T]\) be a closed subinterval of \([t_0, t_1]\), where \(a = nT, \ n \ an \ integer\). Let the coefficient functions \(\alpha(t), \ \gamma(t)\) obey the inequality
\[
\int_a^{a+T} (\gamma(t) - \omega^2 \alpha(t) \sin^2(\omega t)) \ dt > \omega^2 \int_a^{a+T} (\alpha(t) \cos(2\omega t)) \ dt > 0.
\]
Then every solution of (1) will vanish on the interval \([a, a+T]\).

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Proof. The inequality (2) states that
\[ \int_{a}^{a+T} [(\gamma(t) - \omega^2 \alpha(t)) \sin^2 \omega t - \omega^2 \alpha(t) \cos 2\omega t] \, dt > 0. \]
Integrating by parts the second term of the integrand, we have
\[ \int_{a}^{a+T} [(\gamma(t) - \omega^2 \alpha(t)) \sin^2 \omega t + \omega \alpha'(t) \sin(\omega t) \cos(\omega t)] \, dt > 0. \]
We substitute \( u(t) = \sin \omega t \), setting \( \omega^2 = c/a \), where \( c \) and \( a \) are any suitable positive constants. Obviously
\[ (au')' + cu = 0, \]
and \( u(n\pi/\omega) = u((n+1)\pi/\omega) = 0 \) for any integer \( n \). The inequality (3) becomes:
\[ \int_{n\pi/\omega}^{(n+1)\pi/\omega} \left( \left( \gamma(t) - \frac{\alpha(t)\epsilon}{a} \right) u^2 + auu' \left( \frac{\alpha(t)}{a} \right) \right) \, dt \]
\[ = \int_{n\pi/\omega}^{(n+1)\pi/\omega} (u \cdot Lu) \, dt \geq 0. \]
(See for example [2, Equation 1.16, p. 8] for details of manipulation of equality (5).)

The inequality (5) \( \int_{n\pi/\omega}^{(n+1)\pi/\omega} (u \cdot Lu) \, dt \geq 0 \) allows us to apply the classical form of Leighton's variational theorem (see [1]), which concludes that every real solution of (1) will vanish on the interval \([a, a+T]\), completing the proof.

Corollary 1. Theorem 1 is valid if (2) and (5) are replaced by:
\[ (2a) \quad \gamma(t) - \omega^2 \alpha(t) \geq 0 \]
and
\[ (2b) \quad \int_{a}^{a+T} \alpha'(t) \sin 2\omega t \, dt > 0 \]
on \([a, a+T]\), or by the single condition
\[ (2c) \quad \int_{n\pi/\omega}^{(n+1)\pi/\omega} (\gamma(t) \sin^2 \omega t) \, dt \geq \omega^2 \int_{n\pi/\omega}^{(n+1)\pi/\omega} \alpha(t) \cos^2(\omega t) \, dt. \]

Note. (2c) is obtained from (2) after a trigonometric substitution.

Corollary 2. If \( \alpha(t) \in C^1[t_0, \infty) \), \( \gamma(t) \in C[t_0, \infty) \) and condition (2) is satisfied (or if the equivalent conditions (2a), (2b) or the condition (2c) is satisfied), then solutions of (1) are oscillatory, and will vanish on any interval
Example 1. Consider the equation

\[ y'' + \left( vt^m - \frac{1}{2\phi} + K \frac{1 + t}{1 + \sin t} \right) y = 0, \quad 0 \leq t < 3\pi/2, \]

where \( v > 0, \mu \geq 0, \phi > 1 \) and \( K \geq 1 \). We claim that every solution of this equation will vanish on the interval \([0, \pi/2]\). We choose \( \omega = \sqrt{2}/2 \), and \( T = \pi/\omega = \sqrt{2}\pi \). Using inequality (2c), we compute

\[
\int_0^{\pi/2} \left( vt^m - \frac{1}{2\phi} + K \frac{1 + t}{1 + \sin t} \right) \sin^2 \left( \frac{\sqrt{2}}{2} t \right) dt
\]

\[
= \sqrt{2} \int_0^{\pi/2} \left[ \left( \sqrt{2} \xi \right)^m - \frac{1}{2\phi} + K \frac{\sqrt{2} \xi + 1}{\sin(\sqrt{2} \xi) + 1} \right] \sin^2 \xi d\xi
\]

where \( C_{\mu} > 0 \), while \((\sqrt{2} \xi + 1)/(\sin(\sqrt{2} \xi) + 1) \geq \frac{1}{2}\) on the interval \(0 < \xi < \pi\). It follows that the inequality (2c) is satisfied (since \( K \geq 1 \) and \( \phi \geq 1 \)), proving our claim.

Example 2. We claim that any solution of equation \( y'' + K(1/K + \sin t)y = 0, \quad t \geq 0, \) where \( K \) is a real number, vanishes on every interval of length \( \pi \) on the ray \([0, \infty)\). To prove this statement choose \( \omega = 1 \), and check the inequality (2c). We comment that the oscillatory behavior of this equation is well known. (See for example a paper by Elshin [3].) We observe that in the proof of Theorem 1 we have used the assumption that \( a \) and \( c \) (used in the comparison equation (4)) were constant only to facilitate the derivation of inequality (5). However our arguments may be modified as follows:

Let \( t \mapsto \omega(t) = \phi(t) \) be any function of the class \( C^1[t_0, \infty) \), such that \( \phi'(t) > 0, \lim_{t \to \infty} \phi(t) = +\infty \).

We represent \((\phi')^2\) in the form \((\phi')^2(t) = c(t)/a(t)\), and repeat the basic arguments of Theorem 1, as outlined in the proof of Theorem 2.

Theorem 2. If there exists a function \( \phi(t) \in C^2[t_0, t_1] \) such that \( \phi'(t) > 0 \) for all \( t \in [t_0, t_1] \), and \( \sin \phi(0_1) = 0 \) for some \( 0_1, 0_2 \in [t_0, t_1] \), \( 0_2 > 0_1 \geq t_0 \), and such that

\[ \int_{\theta_1}^{\theta_2} \left\{ [y(t) - a(t)(\phi')^2] \sin^2 (\phi(t)) + \phi' \phi' \sin (\phi(t)) \cos (\phi(t)) \right\} dt > 0, \]

then every solution of (1) will vanish on \([0_1, 0_2]\).
proof. We choose a function

\[ a(t) = K \exp \int_-^t - \left( \frac{\phi''(\xi)}{\phi'(\xi)} \right) d\xi, \]

where \( K \) is a positive constant. Clearly \( a(t) \) satisfies the differential equation

\[ a' + (\phi''/\phi')a = 0, \]

and choose

\[ c(t) = a(t)(\phi')^2(t), \quad t_0 \leq t \leq t_1. \]

It is easily checked that \( u(t) = \sin(\phi(t)) \) obeys the differential equation,

\[ (a(t)u')' + c(t)u = 0 \]

and that

\[ u(\theta_1) = u(\theta_2) = 0, \]

while the inequality (6) can be rewritten as:

\[ \int_{\theta_1}^{\theta_2} \left[ \left( \gamma(t) - \frac{c(t)}{a(t)} \right) u^2 + a(t)uu' \left( \frac{a(t)}{a(t)} \right) \right] dt > 0. \]

Now Leighton’s variational theorem can be applied directly, completing the proof. Some obvious corollaries can be obtained by combining this result with the Sturm-Piccone comparison theorem. (See for example [2] for an exposition.)

example 3. We shall use Theorem 2 to demonstrate that the solutions of the equation \( y'' + t^{-1}y' + t^{2r-1}y = 0, \quad t \in (1, \infty), \quad r > 0, \) will vanish on every interval of the form: \( t \in [(n\pi r)^{1/r}, ((n+1)\pi r)^{1/r}], \quad t > 1. \) (It is easy to show that the solutions are oscillatory.)

proof. We choose \( \phi(t) = r^{-1}t^r. \) The original equation can be written in the selfadjoint form \((ty')' + t^{2r-1}y = 0, \) so that \( \alpha(t) = t, \gamma(t) = t^{2r}. \) Choosing \( \theta_1 = (n\pi r)^{1/r}, \quad \theta_2 = [(n+1)\pi r]^{1/r}, \) we compute

\[ \int_{\theta_1}^{\theta_2} \left[ t^{2r} - t(t^{2r-1}) \right] \sin^2(r^{-1}t) + \frac{1}{2}t^{r-1} \sin(2r^{-1}t) \right] dt \]

\[ = \mu + \frac{1}{2} \int_{\theta_1}^{\theta_2} t^{r-1} \sin(2r^{-1}t) dt \]

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\[ = \mu + \frac{1}{2} \int_{\theta_1}^{\theta_2} \sin \eta \eta r = \mu, \]
where \( \mu = \int_0^\theta (r^{2r+1} - r^t) \sin^2(r^{-1}t) \, dt > 0 \). Hence every solution of this equation will vanish on every interval of length \( T = ((n+1)\pi)^{1/r} - (n\pi r)^{1/r} \) for all \( t > 1 \), which was to be shown.

2. The equation

\[
(\alpha(t)x')' + \gamma(t)f(x) = 0, \quad t \geq t_0.
\]

A similar (weaker) result can be obtained more easily for equation (11) or its special case

\[
(\alpha(t)x')' + \gamma(t)K = 0, \quad t \geq t_0.
\]

where \( K \) is an odd integer, and \( \alpha(t) \neq 0 \). (There is no loss of generality in assuming \( \alpha(t) > 0 \).) \( \alpha(t) \in C^2(\theta_0, \theta_1) \), \( \gamma(t) \in C(\theta_0, \theta_1) \). Using the result of this author [4], and putting \( u(t) = \sin \omega t \), \( \alpha = t_1 = n\pi/\omega \), \( \beta = t_2 = (n+1)\pi/\omega \) (using symbolism of [4]), we obtain the following:  

**Corollary.** Let \( G(\xi) \) be any function such that \( G(0) = 0 \), and \( G(\xi) > 0 \) if \( \xi \neq 0 \). Denote \( dG/d\xi \) by \( g(\xi) \). Let \( \omega > 0 \) be a number such that

\[
\int_{t_1}^{t_2} [\gamma(t)G(\sin \omega t) - \omega^2 \alpha(t) \cos^2 \omega t] \, dt > 0
\]

for some integer \( n \). Then any solution \( x(t) \) of equation (12) will have the property that \( |\dot{x}(t)| < (m/K)^{1/K-1} \) for some \( t \in [t_1, t_2] \), \( t_1 = n\pi/\omega \), \( t_2 = (n+1)\pi/\omega \), where \( m = \max_{t \in [t_1, t_2]} (g^2(\sin \omega t)/4G(\sin \omega t)) \), provided such maximum exists.

In the more general case of (11), we easily have a similar result. The inequality (13) with \( G(\xi) \) having identical properties on \( (n\pi/\omega, (n+1)\pi/\omega) \) implies that every solution \( x(t) \) of (11) will have the property that \( f'(x(t)) < m \sum_{t \in [t_1, t_2]} (g^2(\sin \omega t)/4G(\sin \omega t)) \), where as before \( m = \max_{t \in [t_1, t_2]} (g^2(\sin \omega t)/4G(\sin \omega t)) \), provided \( m \) exists.

Example 4. Consider the equation

\[
(x^2y')' + (x^2 + K \sin x)/x y^3 = 0, \quad x > \pi, \quad K \leq 1,
\]

which is equivalent to the Emden-Fowler equation perturbed by the \( (K \sin x)/x \) term.

We claim that all solutions will attain values smaller in absolute value than \( .7 \) on every interval of length equal to \( \pi^2 \). Choosing \( G(\xi) = \xi^2 \) \((m = 1)\), \( \omega = 1/\pi \), we compute according to formula (13)

\[
\int_{n\pi/\omega}^{(n+1)\pi/\omega} \frac{t^2 \sin^2(\omega t)}{4} - \omega^2 \left( t^2 + K \frac{\sin t}{t} \right) \cos^2(\omega t) \, dt
= \int_{n\pi/\omega}^{(n+1)\pi/\omega} \frac{\pi^2}{4} \sin^2 \tau - \cos^2 \tau - \frac{K}{\pi^2} \sin(\pi \tau) \, d\tau.
\]
A rough numerical computation shows that for \( n \geq 1 \), \( \tau \leq 1 \), this integral is positive. Hence \( df(y)/dy = 5y^4 \) will attain values smaller than \( m = 1 \), or \( |y(x)| < \psi(1/5) < .7 \) on some subinterval of \([n\pi^2, (n+1)\pi^2]\), as required.

Clearly this estimate is valid for the Emden-Fowler equation \( y'' + (2/x)y' + y^5 = 0 \), \( x > \pi \). We remark that a more detailed numerical computation would result in an improved estimate.

REFERENCES


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