EXISTENCE OF A SEMINORMAL BASIS IN $C[0, 1]$

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ABSTRACT. The existence of a basis in $C[0, 1]$ consisting of pairwise orthogonal elements is established.

1. Introduction. Let $B$ be a real Banach space, $x \in B$ is orthogonal to $y \in B$, written $x \perp y$, if $\|x + by\| \geq \|x\|$ for all real numbers $b$. $x \in B$ is orthogonal to a subset $S$ of $B$, written $x \perp S$, if $x \perp y$ for all $y \in S$. A basis $\{x_i\}$ of $B$ is normal if, for all $i = 1, 2, \ldots$, $\|x_i\| = 1$ and $x_i \perp [x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots]$, the closed subspace of $B$ spanned by $x_1, x_2, \ldots$. A basis of $B$ is seminormal if $\|x_i\| = 1$ and $x_i \perp x_j$ for all $i \neq j$. Clearly every normal basis is a seminormal basis. The converse of this is not true in general. The problem of existence of a normal basis in $C[0, 1]$ would be solved in the negative if the nonexistence of a seminormal basis in $C[0, 1]$ is proved. However, we will show in this note that $C[0, 1]$ does possess a seminormal basis.

2. Main result. Let $a_0 = 0$, $a_1 = 1$, and if $y' = 2^n$, $a_{j+k} = (2k-1)/(2j)$ ($n = 0, 1, 2, \ldots$; $k = 1, 2, \ldots, 2^n$). We define a sequence $\{x_i\}$ in $C[0, 1]$ as follows:

$$x_0(t) = 1, \quad t = a_0,$$
$$= 0, \quad t \in \{a_1, a_2\},$$
$$= \text{linear, for other } t,$$

$$x_i(t) = 1, \quad t = a_i,$$
$$= 0, \quad t \in \{a_0, a_2\},$$
$$= \text{linear, for other } t,$$

and, for $j = 2^n$ ($n = 0, 1, \ldots$; $k = 1, \ldots, 2^n$),

$$x_{j+k}(t) = 0, \quad t \in \{a_0, a_1, \ldots, a_j, b_j\},$$
$$= 1, \quad t \in \{a_{j+1}, \ldots, a_{j+k}\},$$
$$= -1, \quad t \in \{a_{j+k+1}, \ldots, a_{2j}\} \quad (= \ast, \text{if } k = j),$$
$$= \text{linear, for the other } t.$$
where
\[ b_j = a_{2j+1}, \quad \text{if } n \text{ is even,} \]
\[ = a_{2j}, \quad \text{if } n \text{ is odd.} \]

**Theorem.** \( \{x_i\} \) is a seminormal basis of \( C[0, 1] \).

Obviously, \( \|x_i\| = 1 \). We need only show \( x_i \perp x_j \ (i \neq j) \) and \( \{x_i\} \) is a basis of \( C[0, 1] \).

3. Orthogonality. The fact that \( x_i \perp x_j \ (i \neq j) \) follows from the following:

**Proposition.** Let \( x, y \in C[0, 1] \). Suppose \( \{t \in [0, 1]: |x(t)| - \|x\| = \{t_1, \cdots, t_n\} \). Then \( x \perp y \) if and only if \( y(t_i)x(t_i)y(t_j)x(t_j) = 0 \) for some \( 1 \leq i, j \leq n \).

**Proof.** The condition is clearly sufficient. On the other hand, suppose \( y(t_i)x(t_i)y(t_j)x(t_j) > 0 \) for all \( 1 \leq i, j \leq n \). Then \( y(t_i)x(t_i) \) have the same sign, for all \( i = 1, 2, \cdots, n \). By the continuity of \( x \) and \( y \), there exist positive numbers \( p \) and \( q \), and a closed subset \( F \) of \( [0, 1] \) such that \( \{t_1, \cdots, t_n\} \subseteq F \), \( |x(t)| = \|x\| - p \) for \( t \in [0, 1] \setminus F \), \( |x(t)| = \|x\| - q \) for \( t \in F \), and \( |y(t)| \geq q > 0 \) for \( t \) in \( F \). Hence \( \|x + by\| < \|x\| \) if \( b = -(p/(2\|y\|)) \).

**Corollary.** If \( x \in C[0, 1] \) is such that \( n=1 \) in the Proposition, then \( x \perp y \) if and only if \( y(t_1) = 0 \).

4. \( \{x_i\} \) is a basis. We will show this in two steps:
A. We construct a basis \( \{y_i\} \) of \( C[0, 1] \) from the Schauder basis \( \{z_i\} \) of \( C[0, 1] \).
B. We construct \( \{x_i\} \) from \( \{y_i\} \).

The following method of piecewise construction of new basis in Banach space is important for our purpose here. The proof can be found in [1, p. 64].

**Lemma.** Let \( \{z_n\} \) be a basis of a Banach space \( E \), \( \{m(n)\} \) an increasing sequence of positive integers, \( m(0) = 0 \), and \( \{y_n\} \) a sequence in \( E \) such that
(a) \( \{y_{m(n-1)+1}, \cdots, y_{m(n)}\} = \{z_{m(n-1)+1}, \cdots, z_{m(n)}\} \) (\( n = 1, 2, \cdots \)), and
(b) there exists a positive constant \( C \) such that
\[
\left\| \sum_{i=m(n-1)+1}^{m(n)} c_i y_i \right\| \leq C \left\| \sum_{i=m(n-1)+1}^{m(n)} c_i y_i \right\|
\]
for any sequence \( \{c_i\} \) of real numbers, and \( m(n-1)+1 \leq j \leq m(n) \) (\( n = 1, 2, \cdots \)).

Then \( \{y_i\} \) is a basis of \( E \).
4A. Let \( \{z_j\} \) be the Schauder basis of \( C[0, 1] \), i.e., \( z_0(t) = 1 - t \), \( z_1(t) = t \), and, for \( j = 2^n \) \( (n = 0, 1, \cdots; k = 1, 2, \cdots, 2^n) \),

\[
\begin{align*}
z_{j+k}(t) &= 0, & t \in \{a_0, a_1, \cdots, a_j\}, \\
&= 1, & t = a_{j+k}, \\
&= \text{linear, for the other } t.
\end{align*}
\]

Now define a sequence \( \{y_j\} \) as follows: \( y_0(t) = x_0(t) \), \( y_1(t) = x_1(t) \), and, for \( j = 2^n \) \( (n = 0, 1, \cdots; k = 2, 3, \cdots, 2^n) \),

\[
\begin{align*}
y_{j+1}(t) &= \frac{1}{2}(x_{j+1}(t) + x_{2j}(t)), \\
y_{j+k}(t) &= \frac{1}{2}(x_{j+k}(t) - x_{j+k-1}(t)).
\end{align*}
\]

It is not hard to see the following is true: \( \{y_0, \cdots, y_3\} = \{z_0, \cdots, z_3\} \), and, for \( j = 2^n \) \( (n = 1, 2, \cdots) \),

\[
\{y_j, \cdots, y_{2j+1}\} = \{z_j, \cdots, z_{2j+1}\}, \quad y_i = z_i \quad (i = 2(j + 1), \cdots, 4j - 1).
\]

Now let \( \{c_i\} \) be any sequence of real numbers. Set \( b_i = |c_i| \) \( (i = 0, 1, 2) \) and \( b_3 = |c_3 + \frac{1}{2}c_0| \). Then

\[
\left\| \sum_{i=0}^{3} c_i y_i \right\| = \max\{b_k: 0 \leq k \leq 3\} = \max\{b_k: 0 \leq k \leq j \leq 3\}
\]

\[
= \left\| \sum_{i=0}^{j} c_i y_i \right\| \quad (0 \leq j \leq 3).
\]

And for \( j = 2^n \) \( (n = 1, 2, \cdots) \), set \( b_i = |c_i| \) \( (i = j, j+1, j+2, \cdots, 2j-2, 2j, 2j+1) \) and \( b_{2j+1} = |c_{2j+1} + \frac{1}{2}c_j| \). Then

\[
\left\| \sum_{k=0}^{j+1} c_{j+k} y_{j+k} \right\| = \max\{b_{k+j}: 0 \leq k \leq j + 1\} \geq \max\{b_{k+j}: 0 \leq k \leq m \leq j + 1\}
\]

\[
= \left\| \sum_{k=0}^{m} c_{j+k} y_{j+k} \right\| \quad (0 \leq m \leq j + 1).
\]

Hence by the Lemma cited above, \( \{y_i\} \) is a basis of \( C[0, 1] \).

4B. From the definition of \( x_i \) and \( y_i \), it follows that \( x_i = y_i \) \( (i = 0, 1, 2) \) and \( \{x_{i+1}, \cdots, x_{2j}\} = \{y_{i+1}, \cdots, y_{2j}\} \) \( (j = 2^n, n = 1, 2, \cdots) \). Again let \( \{c_i\} \) be any sequence of real numbers. For \( j = 2^n \) \( (n = 1, 2, \cdots) \) and \( m \) a positive
integer \leq j, let

\[ b_1 = \sum_{i=1}^{m} c_{j+i}, \quad d_1 = \sum_{i=m+1}^{j} c_{j+i}, \]

\[ b_k = -\sum_{i=1}^{k-1} c_{j+i} + \sum_{i=k}^{m} c_{j+i} \quad (k = 2, \ldots, m), \]

\[ d_k = -\sum_{i=m+1}^{m+k-1} c_{j+i} + \sum_{i=m+k}^{j} c_{j+i} \quad (k = 2, \ldots, j - m), \]

then

\[ \left\| \sum_{i=1}^{j} c_{j+i} x_{j+i} \right\| = \max\{|b_1 + d_1|, |b_2 + d_2|, \ldots, |b_m + d_1|, \]

\[ \quad |b_1 + d_1|, |b_2 + d_2|, \ldots, |b_m + d_{j-m}|\} \]

\[ \geq \max\{|b_1 + d_1|, |b_2 + d_1|, \ldots, |b_m + d_1|, |b_1 + d_1|\} \]

\[ \geq \inf\{\max\{|b_1 + r|, |b_2 + r|, \ldots, |b_m + r|, |b_1 + r|\}\} \]

\[ = \frac{1}{2}(\max\{|b_1|, \ldots, |b_m|\} + |B|) \geq \frac{1}{2} \max\{|b_1|, \ldots, |b_m|\} \]

\[ = \frac{1}{2} \left\| \sum_{i=1}^{m} c_{j+i} x_{j+i} \right\|. \]

where \( B = \max\{b_1, \ldots, b_m\} \) if \( \max\{|b_i|: 1 \leq i \leq m\} = -b_k \) for some \( k, 1 \leq k \leq m \); or \( B = \min\{b_1, \ldots, b_m\} \) if \( \max\{|b_i|: 1 \leq i \leq m\} = b_k \) for some \( k, 1 \leq k \leq m \). This completes the proof.

**Reference**


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