SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON A THEOREM OF RUDIN

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Abstract. We give short proofs of a theorem of Rudin about polynomial approximation in $\mathbb{R}^2 \times \Lambda$ and a corollary of this theorem which says that any function algebra on $[0, 1]$ generated by one complex-valued function and $n$ real functions is all continuous functions. At the same time our proof shows that both results hold with $n$ replaced by an arbitrary index set $\Lambda$.

Denote the points of $\mathbb{R}^2 \times \mathbb{R}^\Lambda$ by $(z, t)$. $C(K)$ denotes all continuous functions on $K$.

Theorem 1. Let $K$ be a compact subset of $\mathbb{R}^2 \times \mathbb{R}^\Lambda$ such that $K_t = \{z | (z, t) \in K\}$ does not separate the plane for any $t$. If $f \in C(K)$ and $f_t(z) = f(z, t)$ is analytic at every interior point of $K_t$ then $f$ can be approximated uniformly on $K$ by polynomials in $z$ and $t_a (a \in \Lambda)$.

Proof. Let $A$ be the function algebra on $K$ generated by $z$ and the $t_a$. Let $\mu$ be an extreme point of ball($A^\perp$). Since the closed support of $\mu$ is a set of antisymmetry, $\mu$ is concentrated on some $K_t \times \{t\}$. But by Mergelyan's theorem $f$ can be approximated uniformly there by polynomials in $z$, and so is annihilated of $\mu$. Thus $f \in A$.

The following corollary is immediate from the above but the direct proof is very short.

Theorem 2. Let $K$ be a compact subset of the line. If $f \in C(K)$ and $u_a (a \in \Lambda)$ are real-valued functions in $C(K)$ such that $f$ and the $u_a$ separate the points of $K$ then the function algebra $A$ on $K$ generated by $f$ and the $u_a$ is $C(K)$.

Proof. Again let $\mu$ be an extreme point of ball($A^\perp$). Each $u_a$ must be constant on $S$, the closed support of $\mu$. Thus $f|S$ is a homeomorphism of a compact subset of the line into the plane. It is well known that $f(S)$

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cannot separate the plane so Mergelyan’s theorem shows $\mu = 0$ and thus $A = C(K)$.

**Reference**


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