A FIXED POINT THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

K. GOEBEL¹ AND W. A. KIRK²

Abstract. Let $K$ be a subset of a Banach space $X$. A mapping $F: K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $(k_i)$ of real numbers with $k_i \to 1$ as $i \to \infty$ such that
$$
\|F^i(x) - F^i(y)\| \leq k_i \|x - y\|, \quad x, y \in K.
$$
It is proved that if $K$ is a non-empty, closed, convex, and bounded subset of a uniformly convex Banach space, and if $F: K \to K$ is asymptotically nonexpansive, then $F$ has a fixed point. This result generalizes a fixed point theorem for nonexpansive mappings proved independently by F. E. Browder, D. Göhde, and W. A. Kirk.

In 1965, F. E. Browder [1] and D. Göhde [4] independently proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. This result was also obtained by W. A. Kirk [5], under assumptions slightly weaker in a technical sense, and another proof, more geometric and elementary in nature, has recently been given by K. Goebel [3]. Our purpose here is to extend Browder's result to a more general class of transformations which we shall call "asymptotically nonexpansive" mappings.

A Banach space $X$ is called uniformly convex (Clarkson [2]) if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $\|x\| = \|y\| = 1$ then $\|(x + y)/2\| \leq 1 - \delta(\epsilon)$. In such a space, it is easily seen that the inequalities $\|x\| \leq d$, $\|y\| \leq d$, $\|x - y\| \geq \epsilon$ imply $\|(x + y)/2\| \leq (1 - \delta(e/d))d$. Furthermore, the function $\delta: (0, 2] \to (0, 1]$ may be assumed to be increasing.

Definition. Let $K$ be a subset of a Banach space $X$. A transformation $F: K \to K$ is said to be nonexpansive if for arbitrary $x, y \in K$,
$$
\|Fx - Fy\| \leq \|x - y\|.
$$

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More generally, \( F \) is said to be \textit{asymptotically nonexpansive} if for each 
\( x, y \in K, \)
\[
\| F^i x - F^i y \| \leq k_i \| x - y \|
\]
where \( \{ k_i \} \) is a sequence of real numbers such that \( \lim_{i \to \infty} k_i = 1 \).

It is obvious that for asymptotically nonexpansive mappings it may be 
assumed that \( k_i \leq 1 \) and that \( k_{i+1} \leq k_i \) for \( i = 1, 2, \ldots \), so throughout the 
paper we shall always assume this to be the case.

Our principal result is the following generalization of Browder's 
theorem of [1].

**Theorem 1.** Let \( K \) be a nonempty, closed, convex and bounded subset 
of a uniformly convex Banach space \( X \), and let \( F: K \to K \) be asymptotically 
nonexpansive. Then \( F \) has a fixed point.

**Proof.** For each \( x \in K \) and \( r > 0 \) let \( S(x, r) \) denote the spherical ball 
centered at \( x \) with radius \( r \). Let \( y \in K \) be fixed, and let the set \( R_y \) consist 
of those numbers \( \rho \) for which there exists an integer \( k \) such that

\[
K \cap \left( \bigcap_{i=k}^{\infty} S(F^i y, \rho) \right) \neq \emptyset.
\]

If \( d \) is the diameter of \( K \) then \( d \in R_y \), so \( R_y \neq \emptyset \). Let \( \rho_0 = \text{g.l.b. } R_y \), and
for each \( \epsilon > 0 \) define (cf. [6, p. 411]) \( C_{\epsilon} = \bigcup_{i=1}^{\infty} \left( \bigcap_{k=1}^{\infty} S(F^i y, \rho_0 + \epsilon) \right) \). 
Thus for each \( \epsilon > 0 \) the sets \( C_{\epsilon} \cap K \) are nonempty and convex, so reflexivity
of \( X \) implies that

\[
C = \bigcap_{\epsilon > 0} (C_{\epsilon} \cap K) \neq \emptyset.
\]

Note that for \( x \in C \) and \( \eta > 0 \) there exists an integer \( N \) such that if \( i \geq N, \)
\( \| x - F^i y \| \leq \rho_0 + \eta. \)

Now let \( x \in C \) and suppose the sequence \( \{ F^n x \} \) does not converge to \( x \)
(i.e., suppose \( Fx \neq x \)). Then there exists \( \epsilon > 0 \) and a subsequence \( \{ F^n x \} \)
of \( \{ F^n x \} \) such that \( \| F^n x - x \| \geq \epsilon, i = 1, 2, \ldots \). For \( m > n, \)
\[
\| F^n x - F^m x \| \leq k_m \| x - F^{m-n} x \|,
\]
where \( k_m \) is the Lipschitz constant for \( F^n \) obtained from the definition of
asymptotic nonexpansiveness. Assume \( \rho_0 > 0 \) and choose \( \alpha > 0 \) so that
\[
(1 - \delta(\epsilon/(\rho_0 + \alpha))) (\rho_0 + \alpha) < \rho_0. \]
Select \( n \) so that \( \| x - F^n x \| \geq \epsilon \) and also so that
\( k_m (\rho_0 + \alpha/2) \leq \rho_0 + \alpha. \) If \( N \geq n \) is sufficiently large, then \( m > N \) implies
\[
\| x - F^{m-n} y \| \leq \rho_0 + \alpha/2,
\]
and we have
\[
\| F^n x - F^m y \| \leq k_m \| x - F^{m-n} y \| \leq \rho_0 + \alpha,
\]
\[
\| x - F^m y \| \leq \rho_0 + \alpha.
\]
Thus by uniform convexity of $X$, if $m>N$,
$$
\|(x + F^nx)/2 - F^ny\| \leq (1 - \delta(\epsilon/(\rho_0 + \epsilon)))(\rho_0 + \epsilon) < \rho_0,
$$
and this contradicts the definition of $\rho_0$. Hence we conclude $\rho_0=0$ or $F^nx=\chi$. But $\rho_0=0$ implies $(F^ny)$ is a Cauchy sequence yielding $F^nx=F^nx$ as $n\to\infty$. Therefore the set $C$ consists of a single point which is fixed under $F$.

**Theorem 2.** Under the same assumptions as in Theorem 1, the set $Y$ of fixed points of $F$ is closed and convex.

**Proof.** Closedness of $Y$ is obvious. To show convexity it is sufficient to prove that $z=((x+y)/2 \in Y$ for all $x, y \in Y$. We have
$$
\|F^iz - x\| = \|F^iz - F^ix\| \leq k_i \|z - x\| = \frac{1}{k_i} \|z - x\|
$$
and hence
$$
\|z - F^iz\| \leq \frac{1}{2}(1 - \delta(2/k_i))k_i \|z - y\|
$$
Thus
$$
z = \lim_{i \to \infty} F^iz = \lim_{i \to \infty} F^{i+1}z = F\left(\lim_{i \to \infty} F^iz\right) = Fz.
$$

The following theorem shows that in Theorem 1 it need only be assumed that $F$ is "eventually asymptotically nonexpansive".

**Theorem 3.** Suppose $K$ is a nonempty, closed, bounded and convex subset of a uniformly convex Banach space $X$ and suppose $F:K\to K$ is an arbitrary (even noncontinuous) transformation such that for some integer $n$,
$$
\|F^iz - F^iy\| \leq k_i \|z - y\|, \quad i \geq n,
$$
where $\lim_{i \to \infty} k_i = 1$. Then $F$ has a fixed point.

**Proof.** The transformation $G=F^n$ is asymptotically nonexpansive so it has a nonempty closed and convex fixed point set $Y$. If $x \in Y$ then $Fx=FGx=F^{n+1}x=GFx$ and thus $F:Y\to Y$. Moreover, $F=F^{pn+1}$ on $Y$ for $p=1, 2, \ldots$. Hence
$$
\|Fx - Fy\| = \|F^{n+1}x - F^{n+1}y\| \leq k_{p+1} \|z - y\|, \quad x, y \in Y.
$$
This implies that $\|Fx - Fy\| \leq \|x - y\|, \quad x, y \in Y$, and according to the fixed point theorem for nonexpansive mappings (Browder [1]), $F$ has a fixed point in $Y$.

Finally we show that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.
Example. Let $B$ denote the unit ball in the Hilbert space $l^2$ and let $F$ be defined as follows:

$$F:(x_1, x_2, x_3, \ldots) \rightarrow (0, x_1^2, A_2 x_2, A_3 x_3, \ldots)$$

where $A_i$ is a sequence of numbers such that $0 < A_i < 1$ and $\prod_{i=2}^{\infty} A_i = \frac{1}{2}$. Then $F$ is lipschitzian and $\|Fx - Fy\| \leq 2 \|x - y\|$, $x, y \in B$; and moreover, $\|F^i x - F^i y\| \leq 2 \prod_{j=2}^{i} A_j \|x - y\|$ for $i = 2, 3, \ldots$. Thus

$$\lim_{i \to \infty} k_i = \lim_{i \to \infty} 2 \prod_{j=2}^{i} A_j = 1.$$ 

Clearly the transformation $F$ is not nonexpansive.

References