A FIXED POINT THEOREM FOR ASYMPTOTICALLY
NONEXPANSIVE MAPPINGS

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Abstract. Let $K$ be a subset of a Banach space $X$. A mapping $F: K \rightarrow K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_i\}$ of real numbers with $k_i \rightarrow 1$ as $i \rightarrow \infty$ such that

$$\|F(x) - F(y)\| \leq k_i \|x - y\|, \quad x, y \in K.$$ 

It is proved that if $K$ is a non-empty, closed, convex, and bounded subset of a uniformly convex Banach space, and if $F: K \rightarrow K$ is asymptotically nonexpansive, then $F$ has a fixed point. This result generalizes a fixed point theorem for nonexpansive mappings proved independently by F. E. Browder, D. Göhde, and W. A. Kirk.

In 1965, F. E. Browder [1] and D. Göhde [4] independently proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. This result was also obtained by W. A. Kirk [5], under assumptions slightly weaker in a technical sense, and another proof, more geometric and elementary in nature, has recently been given by K. Goebel [3]. Our purpose here is to extend Browder's result to a more general class of transformations which we shall call "asymptotically nonexpansive" mappings.

A Banach space $X$ is called uniformly convex (Clarkson [2]) if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ then $\|(x+y)/2\| \leq 1 - \delta(\varepsilon)$. In such a space, it is easily seen that the inequalities $\|x\| \leq d$, $\|y\| \leq d$, $\|x - y\| \geq \varepsilon$ imply $\|(x+y)/2\| \leq (1 - \delta(d))d$. Furthermore, the function $\delta: (0,2] \rightarrow (0,1]$ may be assumed to be increasing.

Definition. Let $K$ be a subset of a Banach space $X$. A transformation $F: K \rightarrow K$ is said to be nonexpansive if for arbitrary $x, y \in K$,

$$\|Fx - Fy\| \leq \|x - y\|.$$ 

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More generally, $F$ is said to be \textit{asymptotically nonexpansive} if for each $x, y \in K$,
\[
\|F^nx - F^ny\| \leq k_i \|x - y\|
\]
where $\{k_i\}$ is a sequence of real numbers such that $\lim_{i \to \infty} k_i = 1$.

It is obvious that for asymptotically nonexpansive mappings it may be assumed that $k_i \geq 1$ and that $k_{i+1} \leq k_i$ for $i = 1, 2, \cdots$, so throughout the paper we shall always assume this to be the case.

Our principal result is the following generalization of Browder's theorem of [1].

\textbf{Theorem 1.} Let $K$ be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space $X$, and let $F : K \to K$ be asymptotically nonexpansive. Then $F$ has a fixed point.

\textbf{Proof.} For each $x \in K$ and $r > 0$ let $S(x, r)$ denote the spherical ball centered at $x$ with radius $r$. Let $y \in K$ be fixed, and let the set $R_y$ consist of those numbers $\rho$ for which there exists an integer $k$ such that
\[
K \cap \left( \bigcap_{i=k}^{\infty} S(F^iy, \rho) \right) \neq \emptyset.
\]
If $d$ is the diameter of $K$ then $d \in R_y$, so $R_y \neq \emptyset$. Let $\rho_0 = \text{g.l.b.} R_y$, and for each $\epsilon > 0$ define (cf. [6, p. 411])
\[
C_\epsilon = \bigcup_{i=k}^{\infty} \left( \bigcap_{\rho = \rho_0 + \epsilon} S(F^iy, \rho_0 + \rho_0 + \epsilon) \right).
\]
Thus for each $\epsilon > 0$ the sets $C_\epsilon \cap K$ are nonempty and convex, so reflexivity of $X$ implies that
\[
C = \bigcap_{\epsilon > 0} (C_\epsilon \cap K) \neq \emptyset.
\]
Note that for $x \in C$ and $\eta > 0$ there exists an integer $N$ such that if $i \geq N$,
\[
\|x - F^iy\| \leq \rho_0 + \eta.
\]
Now let $x \in C$ and suppose the sequence $\{F^nx\}$ does not converge to $x$ (i.e., suppose $F^nx \neq x$). Then there exists $\epsilon > 0$ and a subsequence $\{F^nx\}$ of $\{F^nx\}$ such that $\|F^nx - x\| \geq \epsilon$, $i = 1, 2, \cdots$. For $m > n$,
\[
\|F^nx - F^mx\| \leq k \|x - F^{m-n}x\|,
\]
where $k$ is the Lipschitz constant for $F$ obtained from the definition of asymptotic nonexpansiveness. Assume $\rho_0 > 0$ and choose $\alpha > 0$ so that
\[
(1 - \delta(\epsilon(\rho_0 + \alpha)))(\rho_0 + \alpha) < \rho_0.\]
Select $n$ so that $\|x - F^nx\| \geq \epsilon$ and also so that $k_n(\rho_0 + \alpha/2) \leq \rho_0 + \alpha$. If $N \geq n$ is sufficiently large, then $m > N$ implies
\[
\|x - F^{m-n}y\| \leq \rho_0 + \alpha/2,
\]
and we have
\[
\|F^nx - F^ny\| \leq k_n \|x - F^{m-n}y\| \leq \rho_0 + \alpha,
\]
\[
\|x - F^ny\| \leq \rho_0 + \alpha.
\]
Thus by uniform convexity of \( X \), if \( m > N \),
\[
\|(x + F^m x)/2 - F^m y\| \leq (1 - \delta(e/(\rho_0 + \alpha)))(\rho_0 + \alpha) < \rho_0,
\]
and this contradicts the definition of \( \rho_0 \). Hence we conclude \( \rho_0 = 0 \) or \( Fx = x \). But \( \rho_0 = 0 \) implies \( \{F^n y\} \) is a Cauchy sequence yielding \( F^n y \to x = Fx \) as \( n \to \infty \). Therefore the set \( C \) consists of a single point which is fixed under \( F \).

**Theorem 2.** Under the same assumptions as in Theorem 1, the set \( Y \) of fixed points of \( F \) is closed and convex.

**Proof.** Closedness of \( Y \) is obvious. To show convexity it is sufficient to prove that \( z = (x + y)/2 \in Y \) for all \( x, y \in Y \). We have
\[
\|F^i z - x\| = \|F^i z - F^i x\| \leq k_i \|z - x\| = \frac{1}{2} k_i \|x - y\|,
\]
\[
\|F^i z - y\| = \|F^i z - F^i y\| \leq k_i \|z - y\| = \frac{1}{2} k_i \|x - y\|.
\]
Thus
\[
\|z - F^i z\| \leq \frac{1}{2} (1 - \delta(2/k_i)) k_i \|x - y\|
\]
and hence
\[
z = \lim_{i \to \infty} F^i z = \lim_{i \to \infty} F^{i+1} z = F \left( \lim_{i \to \infty} F^i z \right) = Fz.
\]

The following theorem shows that in Theorem 1 it need only be assumed that \( F \) is "eventually asymptotically nonexpansive".

**Theorem 3.** Suppose \( K \) is a nonempty, closed, bounded and convex subset of a uniformly convex Banach space \( X \) and suppose \( F: K \to K \) is an arbitrary (even noncontinuous) transformation such that for some integer \( n \),
\[
\|F^i x - F^i y\| \leq k_i \|x - y\|, \quad i \geq n,
\]
where \( \lim_{i \to \infty} k_i = 1 \). Then \( F \) has a fixed point.

**Proof.** The transformation \( G = F^n \) is asymptotically nonexpansive so it has a nonempty closed and convex fixed point set \( Y \). If \( x \in Y \) then \( Fx = FGx = F^{n+1} x = GFx \) and thus \( F: Y \to Y \). Moreover, \( F = F^{pn+1} \) on \( Y \) for \( p = 1, 2, \ldots \). Hence
\[
\|Fx - Fy\| = \|F^{pn+1} x - F^{pn+1} y\| \leq k_{pn+1} \|x - y\|, \quad x, y \in Y.
\]
This implies that \( \|Fx - Fy\| \leq \|x - y\|, \quad x, y \in Y \), and according to the fixed point theorem for nonexpansive mappings (Browder [1]), \( F \) has a fixed point in \( Y \).

Finally we show that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.
Example. Let $B$ denote the unit ball in the Hilbert space $l^2$ and let $F$ be defined as follows:

$$F: (x_1, x_2, x_3, \ldots) \rightarrow (0, x_1^2, A_2 x_2, A_3 x_3, \ldots)$$

where $A_i$ is a sequence of numbers such that $0 < A_i < 1$ and $\prod_{i=2}^\infty A_i = \frac{1}{2}$. Then $F$ is Lipschitzian and $\|Fx - Fy\| \leq 2 \|x - y\|$, $x, y \in B$; and moreover, $\|F^i x - F^i y\| \leq 2 \prod_{j=2}^i A_j \|x - y\|$ for $i = 2, 3, \ldots$. Thus

$$\lim_{i \to \infty} k_i = \lim_{i \to \infty} 2 \prod_{j=2}^i A_j = 1.$$

Clearly the transformation $F$ is not nonexpansive.

References


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