NONEXISTENCE OF ASYMPTOTIC OBSERVABLES

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Abstract. Strong asymptotic limits of all Heisenberg observables exist only for trivial Hamiltonians.

Let \( H \) be a selfadjoint operator on a separable Hilbert space \( \mathcal{H} \). Lavine [1] has introduced into scattering theory the study of the algebra of bounded operators \( \mathcal{A} \) for which the strong asymptotic limits

\[
\lim_{t \to \pm \infty} e^{-iHt} A e^{iHt}
\]

of the Heisenberg observables \( A(t) = e^{-iHt} A e^{iHt} \) exist. It is therefore of some interest that this algebra coincides with \( \mathcal{B}(\mathcal{H}) \) only in the trivial case.

Theorem. The limit (*) exists for every bounded \( A \) iff \( H \) is a constant multiple of the identity.

Proof. Suppose that \( H \) is not a multiple of \( I \). If \( H \) has two distinct eigenvalues \( \lambda \) and \( \mu \) with eigenvectors \( \phi \) and \( \psi \), choose \( A = (\cdot, \phi) \psi \). Then \( A(t) = e^{i(t–\mu)H}(\cdot, \phi) \psi \) has no limit. Otherwise, \( H \) has a nontrivial continuous part, and since it suffices to construct \( A \) on a reducing subspace of \( H \), one may assume that \( H \) is multiplication by \( \lambda \) on \( L^2([a, b], dg) \) where \( g(\lambda) \) is a continuous increasing function with \( g(a) = 0 \) and \( g(b) = 1 \). If every interval \([\alpha, \beta)\) on which \( g(\lambda) \) is constant is deleted from \([a, b)\), the remaining set supports \( dg \) and is mapped by \( g \) in a one-one measure-preserving fashion onto \([0, 1)\) with Lebesgue measure. Under this change of variables, \( H \) becomes multiplication by the strictly increasing function \( \alpha(x) = g^{-1}(x) \) on \( L^2[0, 1) \). In this representation, choose \( A f(x) = f(1–x) \), so that

\[
A(t) f(x) = e^{-i \beta(x) \lambda} f(1–x)
\]

where \( \beta(x) = \alpha(x) – \alpha(1–x) \) is strictly increasing. Since

\[
\| A(t) f – A(s) f \|^2 = \int_0^1 |1 – e^{i \beta(x)(t–s)}| f(1–x)|^2 dx
\]

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which depends only on $t-s$, the strong limit can exist only if the right side vanishes for all $t$ and $s$. But since $\beta(x)$ is strictly increasing, this implies that $f=0$.

**Reference**


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