THE EXISTENCE OF OSCILLATORY SOLUTIONS FOR THE EQUATION
\[ d^2y/dt^2 + q(t)y^r = 0, \quad 0 < r < 1 \]

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Abstract. This paper gives sufficient conditions for the existence of oscillatory solutions in the sublinear case of the second order differential equation \( d^2y/dt^2 + q(t)y^r = 0 \), where \( q(t) \) is nonnegative and continuous and \( 0 < r < 1 \). We use the technique of [3, Theorem 3.1] and obtain a result which extends [2, Corollary 1], [3, Theorem 3.1], and [3, Theorem 3.2].

We are here concerned with the oscillatory behavior of solutions of the following second order nonlinear differential equation:

\[ (1) \quad d^2y/dt^2 + q(t)y^r = 0 \]

where \( q(t) \geq 0 \) and continuous on \((0, \infty)\) and \( r \) satisfies \( 0 < r = p/q < 1 \) where \( p, q \) are odd integers.

It will be tacitly assumed here that every locally defined solution of (1) is continuously extendable throughout the entire nonnegative real axis. A nontrivial solution \( y(t) \) of (1) is said to be oscillatory if for any positive number \( a \) there exists \( b \) greater than \( a \) such that \( y(b) = 0 \).

For the sake of completeness we state some related results. Belohorec [1] has shown the following result on the existence of one oscillatory solution.

Theorem A. if \( (d/dt)(q(t)t^{(r+3)/2}) \leq 0 \) and \( q(t)t^{(r+3)/2} \geq K_1 > 0 \) for \( t > 0 \), then (1) has oscillatory solutions.

Coffman and Wong [2] obtained a result which is an improvement of the result of Theorem A. namely,

Theorem B. If \( q(t)t^{(r+3)/2}(\log t)^u \) is nonincreasing for some \( u \leq 0 \) and bounded away from 0, then equation (1) has an oscillatory solution.

Very recently, Heidel and Hinton [3] have other results for equation (1), namely,
Theorem C. If \((d/dt)(q(t)t^{(r+3)/2}) \geq 0\) and \(q(t)t^{(r+3)/2} \leq K\) for \(t > 0\), then every solution \(y(t)\) of (1) such that \(y(t_0) = 0, t_0 > 0,\) and \(|y'(t_0)|\) is sufficiently small is oscillatory.

Theorem D. If \(\lim_{t \to \infty} t(d/dt)(q(t)t^{(r+3)/2}) = 0\) and \((d/dt)(q(t)t^{(r+3)/2}) \geq 0\) for \(t > 0\), then every solution \(y(t)\) of (1) such that \(y(t_0) = 0, t_0 > 0,\) and \(|y'(t_0)|\) is sufficiently small is oscillatory.

The purpose of this paper is to show that Theorem C and Theorem D remain valid without the assumption \(q(t)t^{(r+3)/2} \leq K\) for \(t > 0\) in Theorem C, and the assumption \(\lim_{t \to \infty} t(d/dt)(q(t)t^{(r+3)/2}) = 0\) in Theorem D. This will unify Theorems C and D into a single criterion for the existence of oscillatory solutions.

We can now state our theorems which will be proved by refining the technique of [3].

Theorem 1. If \((d/dt)(q(t)t^{(r+3)/2}) \geq 0\) for \(t > 0\), then every solution \(y(t)\) of (1) such that \(y(t_0) = 0, t_0 > 0,\) and \(|y'(t_0)|\) is sufficiently small is oscillatory.

Proof. We make the change of variables, \(x = \log t, y(t) = t^{1/2}w(x),\) which transforms (1) into

\[
\frac{d^2w}{dx^2} - \frac{1}{2}w + f(x)w = 0,
\]

where \(f(x) = q(t)t^{(r+3)/2}\). Clearly, the \((0, \infty)\) \(t\)-interval corresponds to the \((\infty, 0)\) \(x\)-interval. Define \(\beta(x) = (4f(x))(1/(1-r))\) and \(G(w(x))\) by

\[
G(w(x)) = w^2(x)/2 + (f(x)w(x)^{r+1})/(r + 1) - w^2(x)/8.
\]

Then \(G(w(x)) = G(w(x_0)) + \int_{x_0}^{x} f'(u)w(u)^{r+1} du/(r + 1)\).

We claim that if \(w(x_0) = 0\) and \(w^2(x_0)/2 < Kf(x_0)^{2/(1-r)}, K = c^{1+r} \cdot 4^{1/(1-r)}\), \((1-r)/2(1+r), c\) is a constant and \(0 < c < 1\), then \(|w(x)| < c\beta(x)|\) for \(x \geq x_0\). As long as \(|w(x)| \leq c\beta(x)|\), then \(w^2 < c^{r+1} \cdot (4f(x))^{(1+r)/(1-r)}\).

Hence

\[
G(w(x)) \leq w^2(x_0)/2 - Kf(x_0)^{2/(1-r)} + Kf(x)^{2/(1-r)} < Kf(x)^{2/(1-r)}.
\]

Suppose that \(|w(x)| = c\beta(x)|\) for some \(x > x_0\) and let \(x_1 > x_0\) be the first such point. Then

\[
G(w(x_1)) = w^2(x_1)/2 + (f(x_1)c^{r+1}(4f(x_1))^{(1+r)/(1-r)})/(r + 1)
- (c^2(4f(x_1))^{2/(1-r)})/8
\geq w^2(x_1)/2 + Kf(x_1)^{2/(1-r)} \geq Kf(x_1)^{2/(1-r)},
\]

since \(0 < c < 1, 0 < r < 1,\) and \(w^2(x_1)/2 > 0\). But this contradicts \(G(w(x)) < Kf(x)^{2/(1-r)}\) for \(x_0 \leq x \leq x_1\). Therefore, \(|w(x)| < c\beta(x)|\) for \(x \geq x_0\).

Transforming back to \(t\) variables we obtain

\[
(y(t)^{1/2})^{1-r} < c^{1-r}(4q(t)t^{(r+3)/2})\quad \text{for large } t.
\]
Therefore, \( q(t)y(t)^{r-1} > c^{r-1}/(4t^2) = (1+\varepsilon)/(4t^2) \) for some \( \varepsilon > 0 \) and for large \( t \). Thus \( y(t) \) must be an oscillatory solution of \( d^2y/dt^2 + (q(t)y(t)^{r-1})y = 0 \). Therefore, the theorem is proved.

In the proof of the above theorem we use \((d/dt)(q(t)t^{(r+3)/2}) \geq 0\) to show that \( G(w(x)) < Kf(x)^{2/(1-r)} \) for \( |w(x)| \leq c\beta(x) \). We can also assert that this statement is valid if \((d/dt)(q(t)t^{(r+3)/2}) \leq 0 \) and \( q(t)t^{(r+3)/2} \geq K_1 > 0 \) for \( t > 0 \). Therefore, we have the following alternate proof of Belohorec’s result, Theorem A.

**Theorem 2.** If \((d/dt)(q(t)t^{(r+3)/2}) \leq 0 \) and \( q(t)t^{(r+3)/2} \geq K_1 > 0 \) for \( t > 0 \), then every solution \( y(t) \) of (1) such that \( y(t_0) = 0, t_0 > 0, \) and \( |y'(t_0)| \) is sufficiently small is oscillatory.

**Proof.** The proof of this theorem will follow the proof of Theorem 1. In this theorem we may choose \( w^2(x_0)/2 < KK_1^{2/(1-r)} \) where \( K \) is the same as before and \( 0 < K_1 \leq q(t)t^{(r+3)/2}, t > 0 \).

We assert that \( G(w(x)) < Kf(x)^{2/(1-r)} \) as long as \( |w(x)| \leq c\beta(x) \). To see this, consider

\[
G(w(x)) = G(w(x_0)) + \int_{x_0}^{x} (w(u)^{r+1}f(u)(r + 1)) \, du
\leq w^2(x_0)/2 < KK_1^{2/(1-r)} \leq Kf(x)^{2/(1-r)},
\]

since \( f'(u) \leq 0 \) and \( K_1 \leq f(x) \). This proves the assertion.

Then, we may follow the proof of Theorem 1 to obtain the desired result.

**Remark.** In Theorem 1 if we replace \((d/dt)(q(t)t^{(r+3)/2}) \geq 0 \) by the non-decreasing function \( q(t)t^{(r+3)/2} \), then Theorem 1 is still valid. Similarly, in Theorem 2 if we interchange \((d/dt)(q(t)t^{(r+3)/2}) \leq 0 \) and the nonincreasing function \( q(t)t^{(r+3)/2} \), then Theorem 2 is also valid.

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**References**

1. S. Belohorec, *On some properties of the equation \( y''(x) + f(x)y^\alpha(x) = 0 \), 0 < \( \alpha < 1 \), Mat. Časopis Sloven. Akad. Vied. 17 (1967), 10–19. MR 35 #5703.

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