ON CONVEX SUBSETS OF A POLYTOPE

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Abstract. A. J. Hoffman conjectured the following: Given a d-polytope $P$ and a collection, $C_1, \cdots, C_k$, of closed convex subsets of $P$ with the property that each $t$-flat, $0 \leq t \leq d-1$, which meets $P$ also meets some $C_i$, then there exist polytopes $D_1 \subseteq C_i$ such that every $t$-flat which meets $P$ also meets some $D_i$. In this note it is shown that the above is true for $k=2$.

In [2], Hoffman stated the following conjecture $C(d, t, k)$: If $P$ is a d-polytope, $d \geq 1$, and $t \geq 0$ and $k \geq 1$ are integers, $C_1, \cdots, C_k$ are closed convex sets in $P$ such that every $t$-flat that meets $P$ meets $\bigcup_{i=1}^k C_i$, then there are polytopes $D_1, \cdots, D_k$ with $D_i \subseteq C_i$, $1 \leq i \leq k$, such that every $t$-flat which meets $P$ meets $\bigcup_{i=1}^k D_i$ also. He established $C(d, 0, k)$ in this same paper. Zaks ([3], [4], [5]) has shown that $C(d, d-2, k)$, $d \geq 3$, $k \geq 4$, is false, that each of $C(d, d-1, k)$ for all $d$ and $k$, $C(d, t, 1)$ and $C(3, 1, 3)$ is true.

The purpose of this note is to prove $C(d, t, 2)$ for which two lemmas are established and the resulting theorem follows. (For notation, see Grünbaum [1].)

Lemma 1. Let $P$ be a d-polytope and $C_1, \cdots, C_k$ be closed convex subsets of $P$ such that every $t$-flat that meets $P$ meets $\bigcup_{i=1}^k C_i$, $0 \leq t \leq d-2$; then $\text{ske}_m P \subseteq \bigcup_{i=1}^k C_i$, where $m = d - t - 1$.

Proof. Let $F$ be an $m$-face of $P$, and let $H$ be a supporting hyperplane of $P$ such that $H \cap P = F$; let $x$ be an arbitrary point of $F$. The affine flat in $H$, orthogonal to the affine hull of $F$ and passing through $x$, is of dimension $d-1-m = d-1-(d-t-1) = t$; it meets $P$ in exactly $\{x\}$; therefore by the assumption on $C_1, \cdots, C_k$, we have $x \in \bigcup_{i=1}^k C_i$ as promised.

Lemma 2. Suppose $P$ is a d-polytope and $C_1$ and $C_2$ are closed convex subsets of $P$ such that $\text{ske}_m P \subseteq C_1 \cup C_2$; then $P \subseteq C_1 \cup C_2$.

Proof. Let $F \not\subseteq C_1 \cup C_2$, then there is a face $F$ of lowest dimension $m \geq 2$ such that $F \subseteq C_1 \cup C_2$. By minimality of $m$, $\text{bd}(F) \subseteq C_1 \cup C_2$. 

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Let \( p \in \text{rel int}(F) \setminus (C_1 \cup C_2) \) and consider the mapping \( x \mapsto x' \) defined as follows. For \( x \in \text{bd}(F) \), let \( x' \) be the (unique) point of \( \text{bd}(F) \) such that the line \( L(p, x) \) meets \( F \) in the segment \( xx' \). By the choice of \( p \), it follows that \( x \) and \( x' \) are in distinct members of \( \{C_1, C_2\} \). Now

\[
\text{bd}(F) = (\text{bd}(F) \cap C_1) \cup (\text{bd}(F) \cap C_2)
\]

and \( (\text{bd}(F) \cap C_1) \cap (\text{bd}(F) \cap C_2) = \emptyset \) with \( \text{bd}(F) \cap C_1 \) and \( \text{bd}(F) \cap C_2 \) nonempty. Since \( C_1 \) and \( C_2 \) are closed, this contradicts the connectedness of \( \text{bd}(F) \).

**Theorem.** Let \( P \) be a \( d \)-polytope and let \( C_1 \) and \( C_2 \) be closed convex subsets of \( P \) such that every \( t \)-flat, \( 0 \leq t \leq d-1 \), which meets \( P \) also meets \( C_1 \cup C_2 \). Then there exists polytopes \( D_1 \subset C_1, D_2 \subset C_2 \) such that every \( t \)-flat which meets \( P \) meets \( D_1 \cup D_2 \).

**Proof.** From the lemmas, \( P = C_1 \cup C_2 \); hence Hoffman’s theorem in [2] is sufficient to complete the proof.

The portions of \( C(d, t, k) \) which remain open are for \( 0 < t \leq d - 3 \) with \( d \geq 4 \) and \( k \geq 3 \).

**References**

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