A NOTE ON SOME CHARACTERIZATIONS OF SIDON SETS

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Abstract. A simple proof of the following theorem is presented: $E \subset \mathbb{Z}$ is a Sidon set if and only if $A_E(T)$ equals any one of the following three spaces: $U_E(T), \bigcap_{p \geq 1} A_E^p(T),$ and $C_E(T, \omega).$

In this note we present an elementary proof of some characterizations of Sidon sets, announced by K. Ross (joint work with R. E. Edwards and E. Hewitt) at the International Conference on Harmonic Analysis in Maryland, November 1971 (cf. [1]).

We refer to [2] for standard definitions and facts. For $E \subset \mathbb{Z},$ let $U_E(T) = \{ f \in C_E(T) : (S_n(f)) \text{ converges uniformly to } f \},$ and let $A_E^p(T) = \{ f \in C_E(T) : f \in L^p \};$ for a fixed element $\omega \in c_0(E),$ let $A_E(T, \omega) = \{ f \in C_E(T) : \omega f \in L^1 \}. E \subset \mathbb{Z}$ is a Sidon set if $A_E(T) = C_E(T).$ We remark that $E$ is a Sidon set if and only if $\inf \{ \| f \|_{\infty} / \| f \|_A : f \in A_E, f \neq 0 \} > 0.$

Theorem. The following statements are equivalent:

(i) $E$ is a Sidon set,
(ii) $U_E(T) = A_E(T),$
(iii) $\bigcap_{p \geq 1} A_E^p(T) = A_E(T),$
(iv) $A_E(T, \omega) = A_E(T).

Proof. (ii), (iii) $\Rightarrow$ (i): Suppose $E \subset \mathbb{Z}$ is not a Sidon set. Without loss of generality, we assume $E \subset \mathbb{Z}^+.$ By our preceding remark, we select $(f_i),$ a sequence of trigonometric polynomials in $C_E(T),$ subject to the following conditions:

(i) $\| f_i \|_{\infty} < 1/2^i,$
(ii) $\| f_i \|_A = 1/i,$
(iii) $N_i = \min \{ n : n \in \text{support } (f_i) \} > \max \{ n : n \in \text{support } (f_i) \},$ whenever $i > j.$

Set $f = \sum f_i.$ Clearly $f \in C_E,$ and, by (ii), $f \notin A_E.$

By (i), $\langle S_N(f) \rangle$ converges uniformly to $f.$ It follows from (ii) that

$$\| S_N(f) - f \|_{\infty} = \| S_N(f) - (S_N(f) - S_N(f)) - f \|_{\infty} \leq \| S_N(f) - f \|_{\infty} + \| S_N(f) - S_N(f) \|_{\infty} \leq \| S_N(f) - f \|_{\infty} + 1/i, \quad \square$$

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where \( j \) is selected so that \( N_j < N < N_{j+1} \). Therefore, \( f \in U_E \). Since \( \|f_j\|_A \geq \|f_j\|_p \) and \( \sum \|f_j\|_p^p < \infty \) for all \( p > 1 \), we have \( f \in \bigcap_{p > 1} A^p_E \).

(iv) \( \Rightarrow \) (i). Let \( \omega_k \) be so that \( \omega_k < 1/j \) for all \( k \geq i_j \). Select \( f_j \) as above, with the additional requirement that \( N_j \geq i_j \). It easily follows that \( \omega f \in l^1 \). The theorem is proved.

Remark. The proof of (iii), (iv) \( \Leftrightarrow \) (i) can easily be adapted to general compact abelian groups.

References


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