BOUNDARY ZERO SETS OF $A^\infty$ FUNCTIONS SATISFYING GROWTH CONDITIONS

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ABSTRACT. Let $A$ denote the algebra of functions analytic in the open unit disc $D$ and continuous in $D$, and let

$$A^\infty = \{ f \in A : f^{(n)} \in A, n = 0, 1, 2, \ldots \}.$$ 

For $f \in A$ denote the set of zeros of $f$ in $D$ by $Z^0(f)$, and for $f \in A^\infty$ let $Z^\infty(f) = \bigcap_{n=0}^{\infty} Z^0(f^{(n)})$. We study the boundary zero sets of $A^\infty$ functions $F$ satisfying, for some sequence $\{M_n\}$ and some $B > 0$,

$$(1) \quad |F^{(n)}(z)| \leq n! B^n M_n, \quad z \in \overline{D}, \; n = 0, 1, 2, \ldots.$$ 

In particular, when $M_n = \exp(n^p)$, $p > 1$, it is shown that for $E$, a proper closed subset of $\partial D$, there exists $F \in A^\infty$ satisfying (1) and with $Z^\infty(F) = Z^0(F) = E$ if and only if $\int_{\partial D} |\log \rho(e^{i\theta}, E)|^p \, d\theta < +\infty$. Here $\rho(z, E)$ is the distance from $z$ to $E$ and $(1/p) + (1/q) = 1$.

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Before outlining the construction which yields the result stated above, let us recall some known facts and make a few simple observations. If $f \in A$, $f \neq 0$, and satisfies a Lipschitz condition of order $\alpha$, $|f(z) - f(z')| \leq
$C|z-z'|^a$, then $\log |f(z)| \leq a \log \rho(z, Z^a(f)) + \log C$; and, consequently, 
$\int_{-\pi}^\pi \log \rho(e^{i\theta}, Z^a(f)) \, d\theta > -\infty$ by Riesz's theorem. Conversely, Carleson [1] showed that if $E \subset \partial D$ is closed and

$$2) \int_{-\pi}^\pi \log \rho(e^{i\theta}, E) \, d\theta > -\infty,$$

then for any $m>0$ there exists an outer function

$$F \in A^m = \{f \in A : f, f', \ldots, f^{(m)} \in A\}$$

such that $Z^a(F) = Z^a(F') = \cdots = Z^a(F^{(m)}) = E$. This result has been extended to show that there exists an $F$ in $A^\infty$ with $Z^a(F) = Z^\infty(F) = E$ (see [5], [6], or [7]). The extension is also a consequence of a recent theorem of Carleson and S. Jacob, which implies that an outer function $F \in A$ with $|F| \in C_c(\partial D)$ belongs to $A^\infty$.

In case $F$ satisfies the stronger hypothesis (1) we can say more. For, if $F \in A^\infty$, $F \neq 0$, and $E = Z^0(F)$, then it follows from Taylor's formula with remainder that

$$|F(z)| \leq (n!)^{-1} \rho(z, E)^n \max\{|F^{(n)}(z)| : z \in D\}, \quad n = 0, 1, 2, \ldots.$$ 

Thus, because of (1), $|F(z)| \leq \rho(z, E) B^n M_n$, so that

$$-\log |F(e^{i\theta})| \geq \sup\{-n \log \rho(e^{i\theta}, E) - \log B^n M_n : n = 0, 1, 2, \ldots\}.$$ 

The integrability of $\log |F(e^{i\theta})|$ then implies that

$$3) \int_{-\pi}^\pi g^*(\log \rho(e^{i\theta}, E)) \, d\theta < +\infty$$

where $g^*(x) = \sup\{nx - \log B^n M_n : n = 0, 1, 2, \ldots\}$. This was already noted by Carleson [1, p. 330] (with similar proof) in case $M_n = (n!)^a$. See also A. Chollet [2].

It is not to be expected that (3) is, in general, a sufficient condition for the existence of $F \in A^\infty$ satisfying (1) and with $Z^a(F) = Z^\infty(F) = E$. For example, in the case $E = \{1\}$, it is known [4, Theorem 1, equation 6] that the necessary and sufficient condition is

$$4) \int_{-\pi}^\pi h^*(-2 \log \rho(e^{i\theta}, E)) \, d\theta < +\infty$$

where $h^*(x) = \sup\{nx - \log n! B^n M_n : n = 0, 1, \ldots\}$. In particular, if $M_n = n! (\log(n+1))^{kn}$ with $1 < k \leq 2$, then the integral (4) diverges while the integral (3) converges.

Our construction of $A^\infty$ outer functions satisfying a growth condition of form (1) is based on the following theorem. As above, $E$ is a proper
closed subset of $\partial D$ and $\rho(z) = \rho(z, E)$ is the distance from $z$ to $E$. Also, if \((e^{i\alpha_n}, e^{i\beta_n})\) are the complementary arcs of $E$ in $\partial D$, define

$$
\hat{\rho}(\theta) = \frac{1}{2\pi} \left( \frac{1}{\theta - a_n} + \frac{1}{b_n - \theta} \right)^{-1}, \quad \theta \in (a_n, b_n)
$$

$$= 0, \quad e^{i\theta} \in E.
$$

Note that $(4\pi)^{-1} \rho(e^{i\theta}) \leq \hat{\rho}(\theta) \leq \frac{1}{4} \rho(e^{i\theta}) \leq \frac{1}{2}.$

**Theorem 1.** Let $\lambda^*$ be a nonnegative convex infinitely differentiable function such that $\varphi(e^{i\theta}) = \lambda^* (-2 \log \rho(\theta))$ satisfies

(i) $(1/2\pi) \int_{-\pi}^{\pi} |\varphi(e^{i\theta})| \, d\theta \leq M < +\infty$ for some constant $M$;

(ii) $|(d^n/d\theta^n)\varphi(e^{i\theta})| \leq n! K^{n+1} \rho(e^{i\theta})^{-n-1}, \, e^{i\theta} \in \partial D \sim E, \, n=0, 1, 2, \ldots,$

for some constant $K > 0$;

(iii) for every constant $C > 0$, $\varphi(e^{i\theta}) + C \log \rho(e^{i\theta}) \to +\infty$ as $\rho(e^{i\theta}) \to 0$. Then there exists an outer function $F \in A^\infty$ with $Z^0(F) = Z^\infty(F) = E$ and a constant $B > 0$ such that

$$|F^{(n)}(z)| \leq n! B^n \lambda^{(n)}(z), \quad n = 0, 1, \ldots,$$

where $\lambda(n) = \sup\{nx - \lambda^*(x) : x > 0\}$.

**Proof.** Let

$$G(z) = G(z, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \varphi(e^{i\theta}) \, d\theta, \quad z \in D,$$

and let $F = \exp(-G)$. We first assert that the derivatives of $G$ satisfy, for some $K_0 \geq 1$, $|G^{(n)}(z)| \leq n! K_0^{n+1} \rho(z)^{-n-1}, \, n=0, 1, 2, \ldots$. This may be proved by repeating the proof of Lemma 2.3 of [6] and keeping track of the constants which appear there. We omit the details of this computation. In particular, we have the slightly weaker estimate

$$|G^{(n+1)}(z)| \leq n! (2K_0^2)^{n+1} \rho(z)^{-n-2}, \quad n=0, 1, 2, \ldots.$$

Next we claim that

$$|F^{(n)}(z)| \leq n! (4K_0^2)^{n+1} |F(z)| \rho(z)^{-2n}, \quad n = 0, 1, \ldots$$

The proof is by induction on $n$. Now (6) is clear for $n=0$. Assume (6) for $n=0, \ldots, j$. For $n=j+1$,

$$|F^{(j+1)}(z)| = \left| \frac{d^j}{dz^j} F(z) G'(z) \right| \leq \sum_{n=0}^{j} \binom{j}{n} |F^{(j-n)}(z) G^{(n+1)}(z)|$$

$$\leq j! 2^{j-2} (K_0^2)^{j+2} |F(z)| \sum_{n=0}^{j} 2^{j-n} \rho(z)^{-2(n+1-j)-n}. $$
Since \( \rho(z) \leq 2 \), \( 2^{j-n} \rho(z)^{-2(j+1)+n} \leq 2^{j+1} \rho(z)^{-2(j+1)} \). Hence
\[
\sum_{n=0}^{j} 2^{j-n} \rho(z)^{-2(j+1)+n} \leq (j+1) 2^{j+1} \rho(z)^{-2(j+1)},
\]
and (6) follows.

Because \( |F^{(n)}(z)| \leq D_n \rho(z)^{-2n} \) for some constant \( D_n > 1 \),
\[
\log |F^{(n)}(re^{i\theta})| \leq -2n \log \rho(re^{i\theta}) + \log D_n,
\]
and so
\[
\log^+ |F^{(n)}(re^{i\theta})| \leq -2n \log \rho(re^{i\theta}) + \log D_n + 2n \log 2
\]
\[
\leq -2n \log \rho(e^{i\theta}) + \log D_n + 4n \log 2,
\]
where the last inequality follows from \( \rho(e^{i\theta}) \leq 2 \rho(re^{i\theta}) \).

Since \( \log \rho(e^{i\theta}) \) is integrable, \( F^{(n)} \) is of bounded characteristic on \( D \)
(i.e. of class \( N \)). Moreover, the dominated convergence theorem implies that
\[
\lim_{r \to 1} \int_{-\pi}^\pi \log^+ |F^{(n)}(re^{i\theta})| \, d\theta = \int_{-\pi}^\pi \log^+ |F^{(n)}(e^{i\theta})| \, d\theta.
\]

Consequently, \( F^{(n)} \) has the factorization \( B_n S_n H_n \) where \( B_n \) is a Blaschke product,
\( S_n \) is a singular inner function, and \( H_n \) is an outer function for
the class \( N \). See e.g. [3, p. 26]. Thus \( F^{(n)} \) has the bound (5) iff the boundary
values of \( F^{(n)} \) have this bound. By (6),
\[
|F^{(n)}(e^{i\theta})| \leq n! (4K_0^n)^{n+1} |F(e^{i\theta})| \rho(e^{i\theta})^{-2n} \text{ a.e.}
\]

Hence, for some constant \( B > 0 \),
\[
|F^{(n)}(e^{i\theta})| \leq n! B^n |F(e^{i\theta})| \rho(\theta)^{-2n} \text{ a.e.}
\]
or
\[
|F^{(n)}(e^{i\theta})| \leq n! B^n \exp[-2n \log \rho(\theta) - \lambda^*(2 \log \rho(\theta))] \text{ a.e.}
\]
\[
(7)
\]

This establishes (5) and also shows that \( F \in A^\infty \). It is clear from the definition
of \( F \), (iii), and (7) that \( Z^0(F) = Z^\infty(F) = E \).

**Theorem 2.** Let \( E \) be a proper closed subset of \( \partial D \). A necessary and
sufficient condition that there exists \( F \in A^\infty \) with \( Z^0(F) = Z^\infty(F) = E \) and a
constant \( B > 0 \) such that
\[
|F^{(n)}(z)| \leq n! B^n e^{n^p}, \quad n = 0, 1, \ldots,
\]
where \( p > 1 \), is that \( \int_{-\pi}^{\pi} |\log \rho(e^{i\theta})| q \, d\theta < +\infty \), \((1/p) + (1/q) = 1\).

**Proof.** Assuming the existence of such an \( F \), (3) holds with \( g^*(x) = \sup\{nx - n \log B - n^p : n = 0, 1, \ldots\} \). A routine calculation shows \( X^2 = O(g^*(x)) \) for large \( x \). Hence \( \log \rho(e^{i\theta})^q \) is integrable.
For the converse we apply Theorem 1 with \( \lambda^*(x) = (p/q)(x/p)^{a} \). For this \( \lambda^* \), straightforward calculations verify that the hypotheses of Theorem 1 are satisfied and that \( \lambda(n) = n^p \).

Theorem 1 also gives information in some cases when we do not know that (3) is a sufficient condition. For example, the following theorem, due to A. Chollet [2], may be obtained.

**Theorem 3.** Let \( E \) be a proper closed subset of \( \partial D \). If there exists \( F \in \mathcal{A}^\infty \), \( F \neq 0 \), with \( \mathcal{Z}^\infty (F) \supset E \) and a constant \( B \geq 0 \) such that

\[
|F^{(n)}(z)| \leq B^{n}(n!)^{a}, \quad n = 0, 1, \ldots,
\]

where \( a > 1 \), then

\[
\int_{-\pi}^{\pi} \rho(e^{i\theta}, E)^{-1/(a-1)} \, d\theta < +\infty.
\]

In the converse direction, if \( a > 2 \) and (10) holds, then there exists \( F \in \mathcal{A}^\infty \) with \( \mathcal{Z}^\infty (F) = \mathcal{Z}^\infty (F) = E \) and a constant \( B \geq 0 \) such that \( |F^{(n)}(z)| \leq B^{n}(n!)^{2a-1}, \quad n = 0, 1, \ldots \).

**Proof.** If \( F \in \mathcal{A}^\infty \) with \( \mathcal{Z}^\infty (F) \supset E \) satisfies (9), then (3) holds with \( g^*(x) = \sup \{ nx - \log B^{n}(n!)^{a-1} : n = 0, 1, \ldots \} \). Since \( e^{x/(a-1)} = O(g^*(x)) \) for large \( x \), (3) implies (10). In the converse direction apply Theorem 1 with \( \lambda^*(x) = 2e^{-1}(x-1)e^{x/(a-1)} \). Then \( \phi(e^{i\theta}) = 2e^{-1}(x-1)\rho(\theta)^{-1/(a-1)} \) and is easily seen to satisfy (i), (ii), and (iii) of Theorem 1. A simple calculation shows

\[
e^{x(n)} = O(e^{2(a-1)n}(n!)^{2(a-1)}).
\]

**Remark.** Theorem 3 gives another proof that the class of \( \mathcal{A}^\infty \) functions satisfying (9) for \( 1 < a < 2 \) is quasi-analytic.

**Remark.** Mme. Chollet has sharpened the last part of Theorem 3 (unpublished) by showing that the exponent \( 2a - 1 \) may be replaced by \( 2a - 2 \).

**References**


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