BOUNDARY ZERO SETS OF $A^\infty$ FUNCTIONS SATISFYING GROWTH CONDITIONS

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Abstract. Let $A$ denote the algebra of functions analytic in the open unit disc $D$ and continuous in $D$, and let

$$A^\infty = \{ f \in A : f^{(n)} \in A, n = 0, 1, 2, \cdots \}.$$ 

For $f \in A$ denote the set of zeros of $f$ in $D$ by $Z^0(f)$, and for $f \in A^\infty$ let $Z^\infty(f) = \bigcap_{n=0}^{\infty} Z^0(f^{(n)})$. We study the boundary zero sets of $A^\infty$ functions $F$ satisfying, for some sequence $\{M_n\}$ and some $B>0$,

$$(1) \quad |F^{(n)}(z)| \leq n! B^n M_n, \quad z \in \bar{D}, n = 0, 1, 2, \cdots.$$ 

In particular, when $M_n = \exp(n^p)$, $p > 1$, it is shown that for $E$, a proper closed subset of $\partial D$, there exists $F \in A^\infty$ satisfying (1) and with $Z^\infty(F) = Z^\infty(E) = E$ if and only if

$$\int_{\partial D} |\log \rho(\theta, E)|^p d\theta < +\infty.$$ 

Here $\rho(z, E)$ is the distance from $z$ to $E$ and $(1/p) + (1/q) = 1$.

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Before outlining the construction which yields the result stated above, let us recall some known facts and make a few simple observations. If $f \in A, f \neq 0$, and satisfies a Lipschitz condition of order $\alpha$, $|f(z) - f(z')| \leq \cdots$
\[ C|z-z'|^a, \text{ then } \log |f(z)| \leq a \log \rho(z, Z^\theta(f)) + \log C; \text{ and, consequently,} \]
\[ \int_{-\pi}^{\pi} \log \rho(e^{i\theta}, Z^\theta(f)) \, d\theta > -\infty \text{ by Riesz's theorem. Conversely, Carleson [1] showed that if } E \subset \partial D \text{ is closed and} \]
\[ (2) \int_{-\pi}^{\pi} \log \rho(e^{i\theta}, E) \, d\theta > -\infty, \]
then for any \( m > 0 \) there exists an outer function
\[ F \in A^m = \{ f \in A : f, f', \cdots, f^{(m)} \in A \} \]
such that \( Z^\theta(F) = Z^\theta(F') = \cdots = Z^\theta(F^{(m)}) = E \). This result has been extended to show that there exists an \( F \) in \( A^\infty \) with \( Z^\theta(F) = Z^\infty(F) = E \) (see [5], [6], or [7]). The extension is also a consequence of a recent theorem of Carleson and S. Jacob, which implies that an outer function \( F \in A \) with \( |F| \in C^\infty(\partial D) \) belongs to \( A^\infty \).

In case \( F \) satisfies the stronger hypothesis (1) we can say more. For, if \( F \in A^\infty, F \neq 0, \text{ and } E = Z^\infty(F) \), then it follows from Taylor's formula with remainder that
\[ |F(z)| \leq (n!)^{-1} \rho(z, E)^n \max \{|F^{[n]}(z) : z \in D\}, \quad n = 0, 1, 2, \cdots. \]
Thus, because of (1), \( |F(z)| \leq \rho(z, E)^n B^n M_n \), so that
\[ -\log |F(e^{i\theta})| \geq \sup \{-n \log \rho(e^{i\theta}, E) - \log B^n M_n : n = 0, 1, 2, \cdots \}. \]
The integrability of \( \log |F(e^{i\theta})| \) then implies that
\[ (3) \int_{-\pi}^{\pi} g^*(-\log \rho(e^{i\theta}, E)) \, d\theta < +\infty \]
where \( g^*(x) = \sup \{nx - \log B^n M_n : n = 0, 1, 2, \cdots\} \). This was already noted by Carleson [1, p. 330] (with similar proof) in case \( M_n = (n!)^a \). See also A. Chollet [2].

It is not to be expected that (3) is, in general, a sufficient condition for the existence of \( F \in A^\infty \) satisfying (1) and with \( Z^\theta(F) = Z^\infty(F) = E \). For example, in the case \( E = \{1\} \), it is known [4, Theorem 1, equation 6] that the necessary and sufficient condition is
\[ (4) \int_{-\pi}^{\pi} h^*(-2 \log \rho(e^{i\theta}, E)) \, d\theta < +\infty \]
where \( h^*(x) = \sup \{nx - \log n! B^n M_n : n = 0, 1, \cdots\} \). In particular, if \( M_n = n! (\log(n+1))^{kn} \) with \( 1 < k \leq 2 \), then the integral (4) diverges while the integral (3) converges.

Our construction of \( A^\infty \) outer functions satisfying a growth condition of form (1) is based on the following theorem. As above, \( E \) is a proper
closed subset of $\partial D$ and $\rho(z) = \rho(z, E)$ is the distance from $z$ to $E$. Also, if $\{e^{i\alpha_n}, e^{i\beta_n}\}$ are the complementary arcs of $E$ in $\partial D$, define

$$\tilde{\rho}(\theta) = \frac{1}{2\pi} \left( \frac{1}{\theta - a_n} + \frac{1}{b_n - \theta} \right)^{-1}, \quad \theta \in (a_n, b_n)$$

$$= 0, \quad e^{i\theta} \in E.$$  

Note that $(4\pi)^{-1} \rho(e^{i\theta}) \leq \tilde{\rho}(\theta) \leq \frac{1}{4} \rho(e^{i\theta}) \leq \frac{1}{2}.$

**Theorem 1.** Let $\lambda^*$ be a nonnegative convex infinitely differentiable function such that $\varphi(e^{i\theta}) = \lambda^*(-2 \log \tilde{\rho}(\theta))$ satisfies

(i) $(1/2\pi) \int_{-\pi}^{\pi} |\varphi(e^{i\theta})| \, d\theta \leq M < +\infty$ for some constant $M$;

(ii) $|\frac{d^n}{d\theta^n} \varphi(e^{i\theta})| \leq n! K^{n+1} \rho(e^{i\theta})^{-n-1}, \quad e^{i\theta} \in \partial D \sim E, \quad n = 0, 1, 2, \ldots,$

for some constant $K > 0$;

(iii) for every constant $C > 0$, $\varphi(e^{i\theta}) + C \log \rho(e^{i\theta}) \to +\infty$ as $\rho(e^{i\theta}) \to 0$.

Then there exists an outer function $F \in A^\infty$ with $Z^0(F) = Z^\infty(F) = E$ and a constant $B > 0$ such that

$$|F^{(n)}(z)| \leq n! B^n e^{\lambda(n)}, \quad n = 0, 1, \ldots,$$

where $\lambda(n) = \sup \{nx - \lambda^*(x) : x > 0\}$.

**Proof.** Let

$$G(z) = G(z, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \varphi(e^{i\theta}) \, d\theta, \quad z \in D,$$

and let $F = \exp(-G)$. We first assert that the derivatives of $G$ satisfy, for some $K_0 \geq 1$, $|G^{(n+1)}(z)| \leq n! K_0^{n+1} \rho(z)^{-n-1}, \quad n = 0, 1, 2, \ldots$. This may be proved by repeating the proof of Lemma 2.3 of [6] and keeping track of the constants which appear there. We omit the details of this computation. In particular, we have the slightly weaker estimate

$$|G^{(n+1)}(z)| \leq n! (2K_0^2)^{n+1} \rho(z)^{-n-2}, \quad n = 0, 1, 2, \ldots.$$

Next we claim that

$$|F^{(n)}(z)| \leq n! (4K_0^2)^{n+1} |F(z)| \rho(z)^{-2n}, \quad n = 0, 1, \ldots.$$

The proof is by induction on $n$. Now (6) is clear for $n = 0$. Assume (6) for $n = 0, 1, \ldots, j$. For $n = j + 1$,

$$|F^{(j+1)}(z)| = \left| \frac{d^j}{dz^j} F(z)G(z) \right| \leq \sum_{n=0}^{j} \binom{j}{n} |F^{(j-n)}(z)| G^{(n+1)}(z)|$$

$$\leq j! 2^{j-3} (K_0^2)^{j+2} |F(z)| \sum_{n=0}^{j} 2^{j-n} \rho(z)^{-2(j-n)}.$$
Since \( \rho(z) \leq 2 \), \( 2^{j-n} \rho(z)^{-2(j+1)+n} \leq 2^{j+1} \rho(z)^{-2(j+1)} \). Hence

\[
\sum_{n=0}^{j} 2^{j-n} \rho(z)^{-2(j+1)+n} \leq (j + 1) 2^{j+1} \rho(z)^{-2(j+1)},
\]

and (6) follows.

Because \( |F^{(n)}(z)| \leq D_n \rho(z)^{-2n} \) for some constant \( D_n > 1 \),

\[
\log |F^{(n)}(re^{i\theta})| \leq -2n \log \rho(re^{i\theta}) + \log D_n,
\]

and so

\[
\log^+ |F^{(n)}(re^{i\theta})| \leq -2n \log \rho(re^{i\theta}) + \log D_n + 2n \log 2
\]

\[
\leq -2n \log \rho(e^{i\theta}) + \log D_n + 4n \log 2,
\]

where the last inequality follows from \( \rho(e^{i\theta}) \leq 2 \rho(re^{i\theta}) \).

Since \( \log \rho(e^{i\theta}) \) is integrable, \( F^{(n)} \) is of bounded characteristic on \( D \) (i.e. of class \( N \)). Moreover, the dominated convergence theorem implies that

\[
\lim_{r \to 1} \int_{-\pi}^{\pi} \log^+ |F^{(n)}(re^{i\theta})| \, d\theta = \int_{-\pi}^{\pi} \log^+ |F^{(n)}(e^{i\theta})| \, d\theta.
\]

Consequently, \( F^{(n)} \) has the factorization \( B_n S_n H_n \) where \( B_n \) is a Blaschke product, \( S_n \) is a singular inner function, and \( H_n \) is an outer function for the class \( N \). See e.g. [3, p. 26]. Thus \( F^{(n)} \) has the bound (5) iff the boundary values of \( F^{(n)} \) have this bound. By (6),

\[
|F^{(n)}(e^{i\theta})| \leq n! (4K_0^n)^{n+1} |F(e^{i\theta})| \rho(e^{i\theta})^{-2n} \quad \text{a.e.}
\]

Hence, for some constant \( B > 0 \),

\[
|F^{(n)}(e^{i\theta})| \leq n! B^n |F(e^{i\theta})| \rho(\theta)^{-2n} \quad \text{a.e.}
\]

or

\[
|F^{(n)}(e^{i\theta})| \leq n! B^n \exp[-2n \log \rho(\theta) - \lambda^*(-2 \log \rho(\theta))] \quad \text{a.e.}
\]

\[
(7)
\]

This establishes (5) and also shows that \( F \in A^\infty \). It is clear from the definition of \( F \), (iii), and (7) that \( Z^0(F) = Z^\infty(F) = E \).

**THEOREM 2.** Let \( E \) be a proper closed subset of \( \partial D \). A necessary and sufficient condition that there exists \( F \in A^\infty \) with \( Z^0(F) = Z^\infty(F) = E \) and a constant \( B > 0 \) such that

\[
|F^{(n)}(z)| \leq n! B^n e^{nP}, \quad n = 0, 1, \ldots,
\]

where \( p > 1 \), is that \( \int_{-\pi}^{\pi} |\log \rho(e^{i\theta})| \, d\theta < +\infty \), \((1/p) + (1/q) = 1\).

**Proof.** Assuming the existence of such an \( F \), (3) holds with \( g^*(x) = \sup\{nx - n \log B - n^p : n = 0, 1, \ldots\} \). A routine calculation shows \( X^2 = O(g^*(x)) \) for large \( x \). Hence \( |\log \rho(e^{i\theta})| \) is integrable.
For the converse we apply Theorem 1 with $\lambda^*(x) = \frac{p/q}{x/p}$. For this $\lambda^*$, straightforward calculations verify that the hypotheses of Theorem 1 are satisfied and that $\lambda(n) = n^p$.

Theorem 1 also gives information in some cases when we do not know that (3) is a sufficient condition. For example, the following theorem, due to A. Chollet [2], may be obtained.

**Theorem 3.** Let $E$ be a proper closed subset of $\partial D$. If there exists $F \in A^\infty$, $F \not\equiv 0$, with $Z^\infty(F) \supset E$ and a constant $B > 0$ such that

$$|F^{(n)}(z)| \leq B^n(n!)^\alpha, \quad n = 0, 1, \ldots,$$

where $\alpha > 1$, then

$$\int_{-\pi}^{\pi} \rho(e^{i\theta}, E)^{-1/(\alpha-1)} d\theta < +\infty.$$

In the converse direction, if $\alpha > 2$ and (10) holds, then there exists $F \in A^\infty$ with $Z^\infty(F) = Z^\infty(F) = E$ and a constant $B > 0$ such that $|F^{(n)}(z)| \leq B^n(n!)^{2\alpha-1}$, $n = 0, 1, \ldots$.

**Proof.** If $F \in A^\infty$ with $Z^\infty(F) \supset E$ satisfies (9), then (3) holds with $g^*(x) = \sup\{nx - \log B^n(n!)^{\alpha-1} : n = 0, 1, \ldots\}$. Since $e^{x/(x-1)} = O(g^*(x))$ for large $x$, (3) implies (10). In the converse direction apply Theorem 1 with $\lambda^*(x) = 2e^{1-(x-a-1)}$. Then $q(e^{i\theta}) = 2e^{-1}(x-a-1)\rho(\theta)^{-1/(\alpha-1)}$ and is easily seen to satisfy (i), (ii), and (iii) of Theorem 1. A simple calculation shows

$$e^{x(n)} = O(e^{2(x-a-1)}n(n!)^{2\alpha-1}).$$

**Remark.** Theorem 3 gives another proof that the class of $A^\infty$ functions satisfying (9) for $1 < \alpha < 2$ is quasi-analytic.

**Remark.** Mme. Chollet has sharpened the last part of Theorem 3 (unpublished) by showing that the exponent $2\alpha - 1$ may be replaced by $2\alpha - 2$.

**References**


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