ORDINAL SUM-SETS

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Abstract. A finite set, B, of ordinals will be called a sum-set if there are nonzero ordinals \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that the set of sums of \( \alpha_1, \alpha_2, \ldots, \alpha_n \), in all \( n! \) permutations of the summands, is \( B \). Let \( B_k \) denote an arbitrary \( k \)-element sum-set; we consider various matters related to the set of numbers \( n \) for which there are \( n \) summands for \( B_k \).

1. In general, addition of ordinals depends on the order of the summands. Various studies, [1], [3], [4], [5], [6], [7], and [8], have been concerned with determining information about the sets \( E_k \) of natural numbers \( k \) for which there exist \( n \) (not necessarily distinct) ordinals that in all possible orderings yield \( k \) distinct sums. We now investigate the following related problem: We say that a set of \( k \) distinct ordinals \( \beta_1, \beta_2, \ldots, \beta_k \) is a sum-set provided that there are a finite number of (not necessarily distinct) nonzero ordinals \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that the set of sums of \( \alpha_1, \alpha_2, \ldots, \alpha_n \) (in all \( n! \) arrangements) is \( \{\beta_1, \beta_2, \ldots, \beta_k\} \). In this case we say that \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are summands for \( \{\beta_1, \beta_2, \ldots, \beta_k\} \). Thus, here we are concerned with which finite sets of ordinals are sets of sums of a finite number of ordinals.

2. Every nonzero ordinal \( \alpha \) has a unique Cantor normal form:

\[
\alpha = \omega^{\alpha_1} M_1 + \omega^{\alpha_2} M_2 + \cdots + \omega^{\alpha_r} M_r,
\]

where \( r, M_1, M_2, \ldots, M_r \) are nonzero natural numbers and where \( \alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0 \). \( \alpha_1 \) is called the degree of \( \alpha \), written "\( \text{deg}(\alpha) \)", and \( M_1 \) is called the leading coefficient of \( \alpha \).

Every nonzero ordinal \( \alpha \) can be uniquely represented in the form

\[
\alpha = \omega^{\alpha_1} M_1 + \rho,
\]
in which, with reference to \( r, M_1, M_2, \cdots, M_r, \alpha_1, \alpha_2, \cdots, \alpha_r \) in (1),
\[
\rho = 0 \quad \text{if } r = 1;
\]
\[
= \omega^r M_2 + \cdots + \omega^r M_r \quad \text{otherwise}.
\]

We shall refer to (2) as the \textit{remainder form representation of} \( \alpha \); \( \rho \) will be called the \textit{remainder of} \( \alpha \), written "\( \text{rem}(\alpha) \)". Clearly, if \( \rho \neq 0 \), then \( \text{deg}(\rho) < \alpha_1 \). For any set \( A \) of ordinals, let \( R_A = \{ \text{rem}(\alpha) : \alpha \in A \} \).

A set \( A \) is said to be \textit{compatible} if \( A \) is a set \( \{ \alpha_1, \alpha_2, \cdots, \alpha_k \} \) of nonzero ordinals and if each \( \alpha_i, 1 \leq i \leq k \), has the same degree and leading coefficient. If \( A = \{ \alpha_1, \alpha_2, \cdots, \alpha_k \} \) is compatible and if the leading coefficient of each \( \alpha_i \) is \( M \), we say that \( A \) is \( M \)-\textit{compatible} and that \( \text{deg}(\alpha_i) \) is the degree of \( A \), written "\( \text{deg}(A) \)".

Throughout this section \( B \) will always denote a \( k \)-element set of ordinals \( \{ \beta_1, \beta_2, \cdots, \beta_k \}, k = 1, 2, \cdots \); moreover, \( \beta_i = \omega^{\beta_i} M_i + \rho_i \) will be the remainder form representation of \( \beta_i, i = 1, 2, \cdots, k \).

Clearly, every unit set \( B \) is a sum-set. If \( B = \{ \beta_1, \beta_2 \} \) is compatible, we can suppose \( \beta_1 < \beta_2 \). Then \( \beta_1 \) and \( \rho_2 - \rho_1 \) are summands for \( B \).

For any set \( A \), let \( \mathcal{P}^*(A) \) be the set of nonempty subsets of \( A \). If \( J \in \mathcal{P}^*([1, 2, \cdots, n]) \), let \( \Sigma_J \) be the group of permutations of \( J \). In particular, if \( J = \{ 1, 2, \cdots, n \} \), we write "\( \Sigma_n \)" instead of "\( \Sigma_J \)".

Let \( R \) be a set of ordinals. We shall say that \( C \) is an \textit{additive derivative of} \( R \) if there exist ordinals \( \gamma_1, \gamma_2, \cdots, \gamma_m \) for which

\[
C = \left\{ \rho + \sum_{i \in J} \gamma_{(i)} : \rho \in R, J \in \mathcal{P}^*([1, 2, \cdots, m]), \text{ and } \phi \in \Sigma_J \right\}.
\]

**Theorem 1.** \( B \) is a sum-set if and only if for some \( l, M \) satisfying \( 1 \leq l \leq M < \omega_\alpha \).

(i) \( B \) is \( M \)-compatible, and

(ii) \( R_B \) is an additive derivative of \( R_B \) for some \( l \)-element subset \( B_0 \) of \( B \).

**Proof.** Suppose \( \alpha_1, \alpha_2, \cdots, \alpha_n \) are summands for \( B \). Let the remainder form representation of \( \alpha_i \) be \( \alpha_i = \omega^{\deg(\alpha_i)} M_i + r_i \) for \( i = 1, 2, \cdots, n \). Let \( \alpha^* = \max\{\deg(\alpha_i) : 1 \leq i \leq n\} \), let \( A = \{ i : 1 \leq i \leq n \text{ and } \deg(\alpha_i) = \alpha^* \} \), and let \( M = \sum_{i \in A} M_i \). Clearly, \( A \) has \( M \) or fewer elements. For each \( \phi \in \Sigma_n \), let \( j_\phi \) be the largest number \( i \) for which \( \phi(i) \in A \). Then

\[
\sum_{i=1}^{n} \alpha_{\phi(i)} = \omega^{\alpha'} M + r_{j_\phi} + \alpha',
\]

where

\[
\alpha' = 0 \quad \text{if } j_\phi = n,
\]

\[
= \sum_{i=j_\phi+1}^{n} \alpha_{\phi(i)} \quad \text{if } j_\phi < n.
\]
In the latter case, \( \deg(a') < \alpha^* \). Therefore, \( B = \{ \sum_{i=1}^{u} \alpha_{\phi(i)} : \phi \in \Sigma_n \} \) is \( M \)-compatible.

Let \( B_0 = \{ (\alpha^* + r_i) : i \in A \} \); clearly, \( B_0 \) is an \( l \)-element subset of \( B \) for \( 1 \leq l \leq M \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_{m-1} \) be those ordinals among \( \alpha_1, \alpha_2, \ldots, \alpha_n \) for which \( i \notin A \) (if any such exist); let \( \gamma_m = 0 \). According to (3) and (4), \( R_B \) is an additive derivative of \( R_B^* \).

Now suppose that for \( 1 \leq l \leq M < \infty \), conditions (i) and (ii) hold; we can further suppose that \( B_0 = \{ \beta_{u_1}, \beta_{u_2}, \ldots, \beta_{u_l} \} \) and that \( \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m \) are ordinals for which

\[
R_B = \left\{ \rho_{u_i} + \sum_{j \in J} \gamma_{\phi(j)} : 1 \leq i \leq l, J \in \mathcal{P}^*(\{1, 2, \ldots, m\}) \right\}
\]

Surely, \( \gamma_1 = 0 \). Moreover, for each \( J \in \mathcal{P}^*(\{1, 2, \ldots, m\}) - \{ \emptyset \} \) and for each \( \phi \in \Sigma_J \), \( \deg(\sum_{j \in J} \gamma_{\phi(j)}) < \deg(B) \). For \( i = 1, 2, \ldots, l+m-1 \), let

\[
\begin{align*}
\alpha_i &= \omega^{\deg(B)} M + \rho_{u_i} && \text{if } 1 \leq i < l; \\
\alpha_i &= \omega^{\deg(B)} (M - l + 1) + \rho_{u_i} && \text{if } i = l; \\
\alpha_i &= \gamma_i & & \text{if } l < i < l + m.
\end{align*}
\]

Then \( \alpha_1, \alpha_2, \ldots, \alpha_{l+m-1} \) are summands for \( B \).

**Theorem 2.** Every compatible set \( B \) can be extended to a sum-set.

**Proof.** Let \( B \) be \( M \)-compatible. Consider the set \( B^* \) for which \( \omega^{\deg(B)} M, \rho_1, \rho_2, \ldots, \rho_k \) are the summands. Surely, \( B \subseteq B^* \).

3. For each nonzero natural number \( k \), let \( B_k \) denote an arbitrary \( k \)-element sum-set, and let \( N(B_k) \) be the set of natural numbers \( n \) for which there are \( n \) (not necessarily distinct) summands for \( B_k \). For positive natural numbers \( k \) and \( M \), let \( N_{\text{int}}(k, M) = \cap \{ N(B_k) : \text{all } M\text{-compatible } B_k \} \); let \( N_{\text{max}}(k, M) = \max\{ N(B_k) : \text{all } M\text{-compatible } B_k \} \) (if this maximum exists); let \( N_{\text{min}}(k, M) = \min\{ N(B_k) : \text{all } M\text{-compatible } B_k \} \).

**Theorem 3.** For all nonzero natural numbers \( k \) and \( M \), \( N_{\text{max}}(k, M) \) exists and

\[
N_{\text{max}}(k, M) = k + M - 1.
\]

**Proof.** For any \( k, M \), consider \( \beta_i = \omega M + (i-1), i = 1, 2, \ldots, k. \)

Let \( \alpha_i = \omega \) for \( 1 \leq i \leq M \) and, if \( k > 1 \), let \( \alpha_i = 1 \) for \( M + 1 \leq i \leq M + k - 1 \). Then \( \alpha_1, \alpha_2, \ldots, \alpha_{k+k-1} \) are summands for \( \{ \beta_1, \beta_2, \ldots, \beta_k \} \). Thus \( N_{\text{max}}(k, M) \geq k + M - 1 \).

Let \( \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} \) be an arbitrary \( k \)-element \( M \)-compatible set. Let the remainder form representation of \( \gamma_i \) be \( \gamma_i = \omega^* M + \rho_i, i = 1, 2, \ldots, k \), where \( \rho_1 < \rho_2 < \cdots < \rho_k \). Suppose \( \delta_1, \delta_2, \ldots, \delta_L \) are summands
for \( \{\gamma_1, \gamma_2, \cdots, \gamma_k\} \). At least one and at most \( M \) of these \( \delta_i \) are of degree \( \gamma \). For some \( j, 1 \leq j \leq L \), suppose \( \deg(\delta_j) = \gamma \); let \( \rho_j = \text{rem}(\delta_j) \). For any non-zero natural number \( r \), let \( 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_r < \omega^j \). Then the ordinals
\[
\omega^jM + \rho_j, \omega^jM + \rho_j + \sigma_1, \omega^jM + \rho_j + \sigma_2, \cdots, \omega^jM + \rho_j + \sigma_1 + \sigma_2 + \cdots + \sigma_r
\]
are all distinct. Thus there are at most \( k - 1 \) summands for \( \{\gamma_1, \gamma_2, \cdots, \gamma_k\} \) of degree less than \( \gamma \). Consequently, \( L \leq M + k - 1 \), and \( N_{\text{max}}(k, M) = k + M - 1 \).

**Theorem 4.** Let \( M \geq 1 \).

(a) For all \( B_1 \), \( N(B_1) = \{1, 2, \cdots, M\} \);
(b) for all \( B_2 \), \( N(B_2) = \{2, 3, \cdots, M+1\} \);
(c) for \( 3 \leq k \leq M \), \( N_{\text{int}}(k, M) = N(\{\omega^jM + 2^{-i-1} : i = 1, 2, \cdots, k\}) = \{k, k+1, \cdots, M\} \).

**Proof.** For \( k = 1, 2, \cdots \), assume \( M \geq k \); let \( B_k = \{\beta_1, \beta_2, \cdots, \beta_k\} \), where \( \beta_i = \omega^jM + \rho_i \) is the remainder form representation of \( \beta_i, i = 1, 2, \cdots, k \). For \( k \leq j \leq M \), let \( x_i = \omega^jM + \rho_i \) for \( 1 \leq i < j \) and let \( x_j = \omega^jM - (j+1) + \rho_j \). Clearly, for each such \( j \), \( x_1, x_2, \cdots, x_j \) are summands for \( B_j \); hence \( \{k, k+1, \cdots, M\} \subseteq N(B_k) \) for each \( B_k \).

(a) and (b). For \( k = 2, \) assume \( \beta_1 < \beta_2 \). For \( M = 1, 2, \cdots \), let \( \gamma_1 = \gamma_2 = \cdots = \gamma_M = \omega^j + \rho_1 \) and let \( \gamma_{M+1} = \rho_2 - \rho_1 \); then \( \gamma_1, \gamma_2, \cdots, \gamma_{M+1} \) are summands for \( B_2 \). Theorem 3 now guarantees that \( N(B_2) = \{1, 2, \cdots, M\} \) and \( N(B_2) = \{2, 3, \cdots, M+1\} \).

(c). In order to show that for \( 3 \leq k \leq M \), \( N_{\text{int}}(k, M) = \{k, k+1, \cdots, M\} \), it suffices to show that \( N(\{\omega^jM + 2^{-i-1} : i = 1, 2, \cdots, k\}) \subseteq \{k, k+1, \cdots, M\} \)—hence \( \{k, k+1, \cdots, M\} \subseteq N(B_k) \), by the above. Let \( x_1, x_2, \cdots, x_n \) be summands for \( \{\omega^jM + 2^{-i-1} : i = 1, 2, \cdots, k\} \).

Suppose that at least one of the \( x_j \)—say \( x_n \)—is finite. Clearly, at least one of the \( x_j \) must be infinite.

**Case 1.** There are at least two distinct remainders among the infinite \( x_j \). Then one of these must be 1; another is of the form \( 2^l \) for \( l \geq 1 \). Then \( 1 + x_n = 2^l \) for \( m \geq 1 \); therefore \( x_n \) is odd. Also, \( 2^l + x_n = 2^{l+1} \) for some \( p \geq 1 \); therefore \( x_n \) is even. Contradiction!

**Case 2.** All of the infinite \( x_j \) have the same remainder, \( \rho \). Then \( \rho \) must be 1. Let \( j \) be the number of infinite \( x_j \); we can suppose \( x_{j+1} \leq x_{j+2} \leq \cdots \leq x_n < \omega^j \). We must have \( x_{j+1} = 1 \) in order to yield the sum \( \omega^jM + 2 \). Similarly, we must have \( j+2 \leq n \) and \( x_{j+2} \) equal to 2 or 3—in order to yield the sum \( \omega^jM + 4 \). But either of these choices for \( x_{j+2} \) would yield a sum—\( \omega^jM + 3 \) or \( \omega^jM + 5 \)—that is not in \( \{\omega^jM + 2^{-i-1} : i = 1, 2, \cdots, k\} \).

Thus all of the \( x_j \) must be infinite. If \( n > M \), then since each \( x_i \) is of degree 1, the leading coefficient of any sum is at least \( n \)—hence is bigger than \( M \). If \( n < k \), then there are fewer than \( k \) remainders obtained among the \( n! \) permutations.
COROLLARY. If \( \gamma_i = \omega^{-1} M + 2^{-i}, i = 1, 2, \ldots, k, \) and \( k \geq \max(3, M+1) \), then \( \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) is not a sum-set.

4. The determination of \( N_{\text{min}}(k, M) \) is considerably more difficult! It is closely related to problems encountered in [7].

**Theorem 5.** Let \( k, M, \) and \( m \) be arbitrary nonzero natural numbers. Then \( N_{\text{min}}(k, M+m) \leq N_{\text{min}}(k, M) \).

**Proof.** Let \( \beta_i = \omega^\beta M + \rho_i \) be the remainder form representation of \( \beta_i, i = 1, 2, \ldots, k \). Suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are summands for the \( k \)-element set \( \{\beta_1, \beta_2, \ldots, \beta_k\} \). At least one of the ordinals \( \alpha_i, 1 \leq i \leq n \), is of degree \( \beta \); let \( \alpha_o \) be any such ordinal and let the remainder form representation of \( \alpha_o \) be \( \alpha_0 = \omega^\beta (M + \rho_o) \). Let \( \alpha_i^* = \omega^\beta (M + \rho_i) \), and for \( i \neq o \) and \( 1 \leq i \leq n \), let \( \alpha_i^* = \alpha_i \). Let \( \beta_i = \omega^\beta (M + \rho_i), i = 1, 2, \ldots, k \). Then \( \alpha_1^*, \alpha_2^*, \ldots, \alpha_k^* \) are summands for the \( k \)-element set \( \{\beta_1^*, \beta_2^*, \ldots, \beta_k^*\} \).

For each nonzero natural number \( k \) we let

\[
N_{\text{min}}(k) = \min\{N_{\text{min}}(k, M) : M = 1, 2, \ldots\}.
\]

We note that \( N_{\text{min}}(k) = N_{\text{min}}(k, 1) \) for \( 1 \leq k \leq 449 \). (See [3], [4], [5, pp. 265–266], and [7, proofs of Theorems 2, 3, and 4].) Theorems 3, 5, and 6 together imply that for all natural numbers \( k \) and \( M \), except for \( k = 1 \), \( M = 1 \) and for \( k = 2, M = 1 \),

\[
N_{\text{min}}(k, M) < N_{\text{max}}(k, M).
\]

**Theorem 6.** Let \( k \) and \( n \) be nonzero natural numbers. Let

\[
L(n) = \max\{l : (l + 2)^2 - 2(l + 2) - 2L(n) + 1 \leq n \}.
\]

If \( k \leq (L(n) + 2)2^{n-2} - 2L(n) + 1 \), then \( N_{\text{min}}(k) \leq N_{\text{min}}(k, 1) \leq n \).

**Proof.** This follows from [7, Theorem 2].

For example, if \( n = 6 \), then \( L(n) = 2 \); Theorem 6 indicates that for \( k \leq 61 \), \( N_{\text{min}}(k) \leq 6 \).

For each nonzero natural number \( n \), the maximum number \( m_n \) of distinct values that can be assumed by a sum of \( n \) nonzero ordinals in all \( n! \) permutations of the summands has been calculated in [1] and [5]. The numbers \( m_n \) increase with \( n \); for \( n \geq 3, m_n < n! \). It is easily seen from the formulas given by Erdös and Wakučić that for \( n \geq 10, n \neq 14, \)

\[
m_n = 3^{4(k - (1 - 1)) - 3(1 - 1)} 11^{1 - 1} 193^{1 - 1},
\]

where \( n = 5k + l \) for \( k, l \) nonnegative integers with \( l \leq 4 \), and where for nonnegative integers \( r \) and \( s \),

\[
r - s = r - s, \quad r \geq s, \quad 0, \quad r < s.
\]
It immediately follows that if \( k > m_n \), then \( N_{\text{min}}(k) > n \). Moreover, [3], [4], [5], [6], and [7, Theorem 4] yield that for \( 1 \leq k \leq 29 \) and for \( 31 \leq k \leq 449 \), if \( n \) is such that \( m_{n - 1} < k \leq m_n \), then \( N_{\text{min}}(k) = n \).

We note that \( N_{\text{min}}(29) = N_{\text{min}}(31) = 5 \), whereas \( N_{\text{min}}(30) = 6 \); thus \( N_{\text{min}} \) is not monotonic. Furthermore, [8, Theorem 1] indicates that there are infinitely many \( k \) for which there are \( j \) and \( l \) satisfying \( j < k < l \) and \( N_{\text{min}}(j) = N_{\text{min}}(l) < N_{\text{min}}(k) \).

Let \( n \geq 1 \). Let

\[
\mathcal{B}_n = \{ (l_1, r_1, 1), (l_2, r_2, 2), \ldots, (l_n, r_n, n) : r_1 = 0 \text{ and for } i = 1, 2, \ldots, n - 1, l_i \geq 1, \text{ and } r_{i+1} = r_i \text{ or } r_{i+1} = r_i + 1 \}.
\]

For each \( B \in \mathcal{B}_n \) and for each \( j = 0, 1, \ldots, n - 1 \), let \( B_j = \{ (l_i, i) : (l_i, j, i) \in B \} \) and let \( B^j \) be the number of distinct sums \( \sum_i l_i \) such that \( (l_i, j, i) \in B_j \cup B^j \), where \( B_j \) ranges over nonempty subsets of \( B_j \) and \( B^j \) ranges over subsets (possibly empty) of \( \bigcup \{ B_u : u \neq j \} \) and \( 0 \leq u \leq n - 1 \). Let \( B^j = \sum_{i=0}^{n-1} B^j \), let \( C_n = \bigcup \{ B^j : B \in \mathcal{B}_n \} \), and let \( \mathcal{D}_n = \{ 1 + x : x \in C_n \} \).

**Theorem 7.** Let \( k \) and \( n \) be nonzero natural numbers. If there exist nonzero natural numbers \( n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_s \) \((s \geq 1)\) such that \( n = \sum_{i=1}^{s} n_i \) and \( k = \prod_{i=1}^{s} k_i \), where \( k_i \leq n_i \) and for \( i = 2, 3, \ldots, s, k_i \in \mathcal{D}_{n_i} \), then \( N_{\text{min}}(k, n_i) \leq n \).

**Proof.** See the proof of [7, Theorem 1].

**Theorem 8.** Let \( k, m, \) and \( M \) be nonzero natural numbers.

(a) Let \( s \in \mathcal{D}_m \). Then

\[
N_{\text{min}}(sk, M) \leq m + N_{\text{min}}(k, M).
\]

In particular,

\[
N_{\text{min}}(2k, M) \leq 1 + N_{\text{min}}(k, M).
\]

(b) Let \( 1 \leq s \leq m \). Then for some \( M' \), \( M < M' \leq 2M \),

\[
N_{\text{min}}(sk, M') \leq m + N_{\text{min}}(k, M).
\]

**Proof.** This follows from [7, Theorem 3].

**Theorem 9.** Let \( k \) and \( M \) be nonzero natural numbers. Then for some \( M' \), \( M < M' \leq 2M \),

\[
N_{\text{min}}(k + 1, M') \leq 1 + N_{\text{min}}(k, M).
\]

**Proof.** This follows from [7, Theorem 4].
REFERENCES


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