ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PERTURBED LINEAR SYSTEMS

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Abstract. The existence of solutions of the system \( y' + Ay = f(t, y) \) having the form \( y(t) = Z(t)a(t) \) is proved, where \( Z(t) \) satisfies \( Z' + AZ = 0 \) and the vector \( a(t) \) has limit \( a \) as \( t \) increases. Estimates for the rate of convergence to zero of \( a(t) - a \) and of \( y(t) - Z(t)a \) are obtained.

Let \( Z(t) \) be a fundamental matrix of solutions for

\[
Z' + AZ = 0,
\]

where \( z \) is an \( n \)-vector and \( A \) is an \( n \times n \) matrix of constants. We shall be concerned with the possibility of writing solutions of

\[
y' + Ay = f(t, y)
\]

\((y, f—n-vectors)\) in the form

\[
y(t) = Z(t)a(t)
\]

where the vector \( a(t) \) has (finite) limit \( a \) as \( t \to \infty \). Our conditions will be such that both \( a(t) - a \) and \( y(t) - Z(t)a \) converge to zero, and we obtain estimates on the rates of convergence.

Problems of this nature have been investigated ([4]–[7], [10]–[13]) for \( n \)-th order differential equations; of these papers [4], [5], [13] assume that the linear differential equation corresponding to (1) is disconjugate, and are thus able to obtain more precise results. Only [1], [2], [4], [9] seem to have considered the more general question of systems. The present results are related to these; however, we are able to obtain results for a more general class of functions \( f \) (see Corollary 2 and the example preceding it), and our asymptotic estimates are better. The latter is accomplished, in part, by estimating separately each component of \( f \) and by restricting consideration to constant matrices \( A \).

For an \( n \)-vector \( a \) we use the norm \( ||a|| = \sum_{i=1}^{n} |a_i| \), we use also the notations \( |a| = (|a_1|, |a_2|, \cdots, |a_n|) \) and \( 1 = (1, 1, \cdots, 1) \). For two \( n \)-vectors \( p, q \) we write \( p \geq q \) if \( p_1 \geq q_1, p_2 \geq q_2, \cdots, p_n \geq q_n \). For an \( n \times n \)
matrix $B=(b_{ij})$ we shall form an $n$-vector $\|B\|$ with components $\|B\|_i = \sup_{1 \leq i \leq n} |b_{ii}|$, and we shall denote by $|B|$ the matrix $(|b_{ij}|)$. Note that we have $|Ba| \leq \|B\| \cdot \|a\|$. 

**Theorem.** Suppose for a given vector $a$ there exist $T>0$, a fundamental matrix $Z(t)$ of (1), and functions $g, \Psi$ with the following properties:

(i) $f$ is continuous on $[T, \infty) \times \mathbb{R}^n$;
(ii) $g(t, s, x)$ is continuous on $[T, \infty) \times [T, \infty) \times \mathbb{R}^n$, $0 \leq g(t, s, p) \leq g(t, s, q)$ whenever $0 \leq p \leq q$, and $g$ satisfies $|Z^{-1}(t)f(s, y)| \leq g(t, s, |y-Z(s)a|)$ for $(t, s, y) \in [T, \infty) \times [T, \infty) \times \mathbb{R}^n$;
(iii) $\Psi \in C([T, \infty))$, $\Psi>0$, $\lim_{t \to \infty} \Psi(t)=0$, and

$$\Psi(t) \geq \int_t^\infty g(s, s, |Z(s)| \Psi(s)) \, ds$$
on $[T, \infty)$.

Then there exist solutions $y$ of (2) of the form (3) where $\lim_{t \to \infty} a(t)=a$, and we have

$$|a(t) - a| \leq \Psi(t),$$

$$|y(t) - Z(t)a| \leq \int_t^\infty g(s-t, s, |Z(s)| \Psi(s)) \, ds$$
on $[T, \infty)$.

**Proof.** Define an operator $F$ on $\mathcal{A} \equiv \{x(t) \in C([T, \infty)) : |x(t)-a| \leq \Psi(t) \text{ for } t \geq T\}$ by

$$Fx(t) = a - \int_t^\infty Z^{-1}(s)f(s, Z(s)x(s)) \, ds;$$

we shall use the Schauder-Tychonov theorem [2] to prove that $F$ has a fixed point in $\mathcal{A}$. Since for $x \in \mathcal{A}$

$$\left| \int_t^\infty Z^{-1}(s)f(s, Z(s)x(s)) \, ds \right| \leq \int_t^\infty g(s, s, |Z(s)[x(s)-a]|) \, ds \leq \int_t^\infty g(s, s, |Z(s)| \Psi(s)) \, ds \leq \Psi(t),$$

$F$ is indeed defined on $\mathcal{A}$ and maps $\mathcal{A}$ into $\mathcal{A}$. It is easily seen in a similar manner that $F$ is continuous, i.e., if $x_n \in \mathcal{A}$ and $x_n \to x$ uniformly on compact subsets of $[T, \infty)$, then $F x_n \to Fx$ uniformly on compact subsets of $[T, \infty)$. Since the functions in $\mathcal{A}$ are bounded, it remains only to show that the functions of $F \mathcal{A}$ are equicontinuous. Let $x \in \mathcal{A}$ and $t_1 \geq t_2 \geq T$; then

$$|Fx(t_1) - Fx(t_2)| \leq \int_{t_2}^{t_1} |Z^{-1}(s)f(s, Z(s)x(s))| \, ds \leq \int_{t_2}^{t_1} g(s, s, |Z(s)| \Psi(s)) \, ds,$$

which is small if $|t_1-t_2|$ is since $g$ is continuous.
Let $a(t) \in \mathcal{A}$ be a fixed point of $F$; then $y(t) = Z(t)a(t)$ is a solution of (2) on $[T, \infty)$, and (4) is established. To prove (5) observe that

$$|y(t) - Z(t)a| = \left|Z(t) \int_t^\infty Z^{-1}(s)f(s, Z(s)a(s)) \, ds\right|$$

$$\leq \int_t^\infty |Z^{-1}(s-t)f(s, Z(s)a(s))| \, ds$$

$$\leq \int_t^\infty g(s-t, s, |Z(s)|\psi(s)) \, ds.$$

Two subcases of this theorem are of particular interest, and we discuss them as corollaries. The first deals with the case where $|Z^{-1}(s)f(t, Z(t)a(t))|$ is integrable for any bounded continuous $a$.

**Corollary 1.** Suppose for some fundamental matrix $Z(t)$ of $\mathcal{A}$ there exists a constant $M$ and a continuous function $h$ on $[0, \infty)^{n+2}$ such that

(i) $0 \leq h(t, s, p) \leq h(t, s, q)$ whenever $0 \leq p \leq q$;

(ii) $|Z^{-1}(t)f(s, y)| \leq h(t, s, |y|)$;

(iii) $\int_t^\infty h(s, s, M\|Z(s)\|) \, ds < \infty$.

Then for all $a \in \mathbb{R}^n$ such that $\|a\| < \frac{1}{2}M$ the hypotheses of the theorem are satisfied with

$$g(t, s, p) \equiv h(t, s, p + \frac{1}{2}M\|Z(s)\|),$$

$$\psi(t) \equiv \int_t^\infty g(s, s, \frac{1}{2}M\|Z(s)\|) \, ds.$$

**Proof.** It is necessary to verify (ii) and (iii) of the theorem. But

$$|Z^{-1}(t)f(s, y)| \leq h(t, s, |y - Z(s)a| + \|Z(s)\| \cdot \|a\|) \leq g(t, s, |y - Z(s)a|),$$

verifying (ii). Choosing $T$ so large that $\int_T^\infty h(s, s, M\|Z(s)\|) \, ds < (M/2n)1,$ and observing that $g$ is increasing in its last argument, we have

$$\psi(t) = \int_t^\infty g(s, s, \frac{1}{2}M\|Z(s)\|) \, ds = \int_t^\infty h(s, s, M\|Z(s)\|) \, ds < \frac{M}{2n}1,$$

so $\|\psi(t)\| < \frac{1}{2}M$, and

$$\int_t^\infty g(s, s, |Z(s)|\psi(s)) \, ds \leq \int_t^\infty g(s, s, \frac{1}{2}M\|Z(s)\|) \, ds = \psi(t).$$

We remark that the error estimate (5) can be replaced by the weaker but more convenient statement

$$|y(t) - Z(t)a| \leq \int_t^\infty g(s-t, s, \frac{1}{2}M\|Z(s)\|) \, ds.$$
This corollary is connected with some results of [2], [4], [5], [7], [11], [12] when specialized to the \( n \)th order linear differential equation

\[
\dot{x}^{(n)} = f(t, x, x', \ldots, x^{(n-1)})
\]

where \( f \) satisfies \(|f(t, x, \ldots, x^{(n-1)})| \leq \sum_{i=0}^{n-1} g_i(t) |x^{(i)}|^{r_i} \) with \( r_i > 0 \) and

\[
\int_0^\infty t^{n-1} (1 + t + \cdots + t^{n-i-1})^{r_i} g_i(t) \, dt < \infty \quad (i = 0, \ldots, n - 1).
\]

Indeed, writing (7) as a first-order system in the usual way (cf. [2]), we obtain the result that for any \( \alpha \) there exists a solution \( x_\alpha \) of (7) such that

\[
|x_\alpha^{(j)} - \alpha_{j+1} - t\alpha_{j+2} - \cdots - (t^{n-j-1}/(n - j - 1)!)| \leq \int_t^\infty (s - i) \left( \sum_{i=0}^{n-1} \|\alpha\|^{r_i} (1 + s + \cdots + s^{n-i-1})^{r_i} g_i(s) \right) \, ds
\]

\( (j = 0, \ldots, n - 1) \).

We turn now to a different application of our main result. Consider the example of (2) given by the single first-order equation

\[
y' = f(t, y) \equiv (1/t)(y - 1)^2 + 1/t^3;
\]

with \( \alpha = 1, \ Z(t) = 1 \), we have \( g(t, s, \|y - \alpha\|) = (1/t)|y - 1|^2 + 1/t^3 \). Since \( (1/t)|y(t) - 1|^2 \) is not necessarily integrable even when \( \lim_{t \to \infty} y(t) = 1 \), Corollary 1 is not applicable, and neither are the results of [2], [4]. However, our main theorem can be applied with \( \psi(t) = t^{-3+\varepsilon} \) for any \( \varepsilon > 0 \), as is readily seen, and from (5) we conclude that there exist solutions \( y(t) \) of (9) such that \( |y(t) - 1| \leq 1/t^2 \). In this example the right-hand side \( f \) has the form

\[
f(t, y) = f_1(t, y - Z(t)\alpha) + f_2(t, y),
\]

where \( f_2 (-t^{-3}) \) is integrable for bounded \( y \) but \( f_1 \) is not integrable even for all \( y \) tending to \( Z(t)\alpha \). We generalize this situation in the following corollary.

**Corollary 2.** Suppose for some fundamental matrix \( Z(t) \) of (1) and some \( \alpha \) we have (10) valid, where there exist continuous functions \( h_1, h_2 \) on \( [0, \infty) \) satisfying

(i) \( 0 \leq h_i(t, s, p) \leq h_i(t, s, q) \) if \( 0 \leq p \leq q \) \( (i = 1, 2) \);

(ii) \( |Z^{-1}(t)f_2(s, y)| \leq h_2(t, s, |y|) \), with

\[
\int_0^\infty h_2(s, \|Z(s)\| \cdot \|\alpha\|) \, ds < \infty;
\]

(iii) \( |Z^{-1}(t)f_1(s, y)| \leq h_1(t, s, |y - Z(s)\alpha|) \).
Suppose finally that there exists a continuous function \( \Phi(t) > 0 \) satisfying, for some \( \epsilon > 0 \) and all large \( t \), \( \Phi(t) \to 0 \) as \( t \to \infty \),

\[
\Phi(t) \geq \int_t^\infty h_1(s, s, (1 + \epsilon) |Z(s)| \Phi(s)) \, ds,
\]

\[
\int_t^\infty h_2(s, s, 2 \|Z(s)\| \cdot \|\alpha\|) \, ds \leq \epsilon \Phi(t).
\]

Then the hypotheses of the theorem are satisfied for large \( T \) with

\[
g(t, s, p) = h_1(t, s, p) + h_2(t, s, p + \|Z(s)\| \cdot \|\alpha\|),
\]

\[
\psi(t) = \Phi(t) + \int_t^\infty h_2(s, s, 2 \|Z(s)\| \cdot \|\alpha\|) \, ds.
\]

**Proof.** Hypothesis (ii) of the theorem is clearly satisfied; we have only to verify (iii). But for sufficiently large \( t \),

\[
\int_t^\infty g(s, s, |Z(s)| \psi(s)) \, ds \leq \int_t^\infty h_1(s, s, (1 + \epsilon) |Z(s)| \Phi(s)) \, ds
\]

\[
+ \int_t^\infty h_2(s, s, |Z(s)| \psi(s) + \|Z(s)\| \cdot \|\alpha\|) \, ds
\]

\[
\leq \Phi(t) + \int_t^\infty h_2(s, s, 2 \|Z(s)\| \cdot \|\alpha\|) \, ds = \psi(t).
\]

As an application of Corollary 2 we determine conditions on the coefficients of the differential equation

\[
u^{(n)} = \sum_{i=0}^{n-1} a_i(t)(u^{(i)})r_i + b(t)
\]

\((r_i > 1)\) which guarantee the existence of a solution \( u(t) \) such that \( u^{(j)}(t) \to 0 \) as \( t \to \infty \) \((j=0, 1, \cdots, n-1)\). If \( b \) and all the \( a_i \) are integrable, this follows from Corollary 1, hence we assume that not all \( a_i \) are integrable. Writing the equation as a system in the usual manner, we have

\[
Z(t) = \begin{bmatrix}
1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)!
0 & 1 & t & \cdots & t^{n-2}/(n-2)!
& \ddots & \ddots & \ddots & \vdots
& & \ddots & \ddots & \ddots
& & & \ddots & \ddots & \ddots
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]
for fundamental matrix and \( \alpha = 0 \). With

\[
h_{2,k}(t, s, |y|) = \frac{t^{n-k}}{(n-k)!} |b(s)|
\]

\( k = 1, \ldots, n \),

\[
h_{1,k}(t, s, |y|) = \frac{t^{n-k}}{(n-k)!} \sum_{i=0}^{n-1} |a_i(s)| |y_{i+1}|^r_i
\]

we have that \( \phi \) must satisfy

\[
\phi_k(t) \geq \left( 1 + \varepsilon \right)^r \int_t^\infty \frac{s^{n-k}}{(n-k)!} \sum_{i=0}^{n-1} |a_i(s)| \left[ \frac{C_i}{s^{i-1}} \phi_i(s) \right]^r_i ds.
\]

For convenience let \( C_i = \sum_{j=1}^{n-1} (n-1)!/(j-1)! (n-j)! \) and \( r = \max r_i \). Let \( \phi_1(t) \) be any solution of

\[
\phi_1(t) \geq (1 + \varepsilon)^r \int_t^\infty \frac{s^{n-1}}{(n-1)!} \sum_{i=0}^{n-1} |a_i(s)| \left[ \frac{C_i}{s^{i-1}} \phi_i(s) \right]^r_i ds
\]

and define

\[
\phi_j(t) = \frac{(n-1)!}{(n-j)! t^{j-1}} \phi_1(t) \quad (j = 2, \ldots, n).
\]

Then

\[
\phi_j(t) \geq (1 + \varepsilon)^r \int_t^\infty \frac{s^{n-j}}{(n-j)!} \sum_{i=0}^{n-1} |a_i(s)| \left[ \frac{C_i}{s^{i-1}} \phi_i(s) \right]^r_i ds
\]

\[
\geq \left( 1 + \varepsilon \right)^r \int_t^\infty \frac{s^{n-j}}{(n-j)!} \sum_{i=0}^{n-1} |a_i(s)| (1 + \varepsilon)^r_i \left[ \frac{C_i}{s^{i-1}} \phi_i(s) \right]^r_i ds,
\]

as required. It thus remains to show that (12) has a suitable solution. Suppose \( 0 \leq \phi_1(t) \leq 1 \), and let \( q = \min r_i > 1 \), so \( \phi_1 \leq \phi^2_1 \); then if \( \phi_1 \) satisfies

\[
\phi_1(t) \geq \int_t^\infty \frac{(1 + \varepsilon)^r s^{n-1}}{(n-k)!} \sum_{i=0}^{n-1} |a_i(s)| \left( \frac{C_i}{s^{i-1}} \phi^2_i(s) \right) ds \equiv \int_t^\infty k(s) \phi^2(s) ds,
\]

it will also satisfy (12). The equality in (13) can be solved to yield

\[
\phi_1(t) = \left[ \phi_1(t_0) \right]^{1-q} + (q - 1) \int_{t_0}^t k(s) ds
\]

provided \( \int_t^\infty k(s) ds = \infty \) and the positive constant \( \phi_1(t_0) \) is chosen less than 1. Thus (11) has solutions tending to zero with their first \( n-1 \) derivatives if

\[
\int_{t_0}^t \sum_{i=0}^{n-1} |a_i(s)| s^{r_i(1-1)+n-1} ds \to \infty
\]

as \( t \to \infty \) and \( b(t) \) satisfies \( \int_t^\infty s^{n-1} |b(s)| ds \leq \varepsilon (n-1)! \phi_1(t) \) for some \( \varepsilon > 0 \).

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REFERENCES


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