SURFACES WITH MAXIMAL LIPSCHITZ-KILLING CURVATURE IN THE DIRECTION OF MEAN CURVATURE VECTOR

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Abstract. $M$ is an oriented surface in $E^{2+N}$. If $M$ is pseudo-umbilical, the Lipschitz-Killing curvature takes maximum in the direction of mean curvature vector. The converse is also investigated. Furthermore assuming that $M$ is closed, pseudo-umbilical and its Gaussian curvature has some nonnegative lower bound, $M$ is completely determined by the $M$-index of $M$.

1. Let $M$ be an oriented Riemannian surface with an isometric immersion $x:M^2\rightarrow E^{2+N}$ in a euclidean space $E^{2+N}$. Let $F(M)$ and $F(E^{2+N})$ be the bundles of orthonormal frames of $M$ and $E^{2+N}$ respectively. Throughout this paper we assume that the mean curvature vector $H$ of $M$ is nowhere zero. Let $B$ be the set of elements $b=(p, e_1, e_2, \ldots, e_{2+N})$ such that $(p, e_1, e_2) \in F(M)$, $e_3=H/|H|$ and that $(x(p), e_1, e_2, e_3, \ldots, e_{2+N}) \in F(E^{2+N})$ whose orientation is coherent with that of $E^{2+N}$, identifying $e_2$ with $dx(e_2)$, $i=1, 2$. Let $\tilde{x}:B\rightarrow F(E^{2+N})$ be the mapping naturally defined by $\tilde{x}(b)=(x(p), e_1, \ldots, e_{2+N})$.

We have the differential forms $\omega_i$, $\omega_{ij}$, $\omega_{ia}$, $\omega_{i\beta}$ $(1\leq i, j\leq 2, 3\leq \alpha, \beta\leq 2+N)$ on $B$ derived from the basic forms and the connection forms on $F(E^{2+N})$ through $\tilde{x}$ as follows.

$$dx = \omega_1 e_1 + \omega_2 e_2,$$
$$de_A = \sum_{B=1}^{2+N} \omega_{AB} e_B,$$
$$\omega_{AB} = -\omega_{BA},$$
$$(A, B = 1, 2, \cdots, 2+N);$$
$$\omega_{ia} = \sum_{j=1}^{2} A_{sij} \omega_j,$$
$$A_{sij} = A_{sji}.$$

In the following, for the summation notations $\sum_i$, $\sum_\alpha$ and $\sum_r$ we mean $\sum_{i=1}^2$, $\sum_{\alpha=3}^{2+N}$ and $\sum_{r=3}^{2+N}$, for the indices $i, j, \alpha, \beta, r, t$ we mean $1\leq i, j\leq 2$, $3\leq \alpha, \beta\leq 2+N$, $4\leq r, t\leq 2+N$. 

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Now we choose \( e_1, e_2 \) as the principal directions of \( e_3 \), then with respect to the frame \((e_1, e_2, e_3, \cdots, e_{2+\lambda})\) the matrices \( A_x = (A_{xij}) \) are written in

\[
(A_{xij}) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad (A_{rij}) = \begin{pmatrix} c_r & d_r \\ d_r & -c_r \end{pmatrix}.
\]

\( H = \frac{1}{2}(a+b)e_3 \). When \( a=b \), \( M^2 \) is pseudo-umbilical. Let \( e \) be a unit normal vector to the tangent plane \( dx(T_p(M^2)) \) at \( x(p) \). Then

\[
e = \sum_a \xi_a e_a, \quad \sum_a \xi_a^2 = 1.
\]

Let \( A(e) \) be the following matrix

\[
A(e) = \sum_a \xi_a A_a = \begin{pmatrix} a \xi_a \xi_a & \sum_r c_r \xi_r \\ \sum_r d_r \xi_r & b \xi_3 \xi_3 - \sum_r c_r \xi_r \end{pmatrix}.
\]

The Lipschitz-Killing curvature \( G(p, e) \) is given by [4]

\[
G(p, e) = \det(A(e))
\]

\[
= ab \xi_3^2 + (b-a) \xi_3 \sum_r c_r \xi_r - \left( \sum_r c_r \xi_r \right)^2 - \left( \sum_r d_r \xi_r \right)^2.
\]

Let \( S_2 \) be the set of all real symmetric square matrices of order 2. Let \( m:S_2 \to \mathbb{R} \) be a linear transformation defined by [6] \( m(A) = \frac{1}{2} \) trace \( A \), \( A \in S_2 \). We denote the normal space to \( x(M^2) \) at \( x(p) \) by \( N_p \), \( N_p = \{ X, X = \sum_a \eta_a e_a, \eta_a \in \mathbb{R} \} \), and define a linear mapping \( \tilde{m}:N_p \to \mathbb{R} \) by

\[
\tilde{m}(X) = \sum_a \eta_a m(A_a), \quad X = \sum_a \eta_a e_a.
\]

The kernel of \( \tilde{m} \) is denoted by \( \ker \tilde{m} \).

At any point \( p \in M^2 \) we take a frame \( b=(p, e_1, \cdots, e_{2+\lambda}) \in B \). Let \( \psi_b:N_p \to S_2 \) be the linear mapping defined by

\[
\psi_b \left( \sum_a \eta_a e_a \right) = \sum_a \eta_a A_a.
\]

The dimension of \( \psi_b(\ker \tilde{m}) \) is called the \( M \)-index of \( M^2 \) at \( p \) and is denoted by \( M \)-index, \( M^2 \) [6].

2. We prove the following lemma.

**Lemma.** If at any point \( p \in M^2 \) the Lipschitz-Killing curvature \( G(p, e) \) has maximum in the direction of \( H \) then \( ab \geq 0 \) where \( a, b \) are given by (1).
PROOF. $e_3 = H || H |$, which is given by $\xi_3 = 1, \xi_r = 0$ in (2). By (2) and (3) it is easy to see that if $G(p, e)$ takes maximum at $\xi_3 = 1, \xi_r = 0$ then $(a-b)c_r = 0$. Hence by (3) the maximum of $G(p, e)$ is $ab$. Now let $S_p$ be an arbitrary chosen unit circle in $N$, and $e'$ be a fixed point in $S_p$. Put $S_p^* = S_p - \{e'\}$. We choose $e \in S_p^*$ and $e_3(e), e_2(e)$ as two unit orthogonal tangent vectors in the principal directions of $e$ and move $e$ differentiably on $S_p^*$. Then the principal curvatures $k_1(e)$ and $k_2(e)$ with respect to $e_1(e)$ and $e_2(e)$ are continuous on $S_p^*$. Now suppose $k_1(e) \neq k_2(e)$ at some $e \in S_p^*$. Then by the continuity of $k_1$ on $S_p^*$ and the fact $k_1(-e) = -k_1(e)$ we see that $k_1 = 0$ for some points in $S_p^*$. This implies that the Lipschitz-Killing curvature $G(p, e) = 0$ for some $e \in S_p^*$. Since $ab$ is the maximum of $G(p, e)$ we conclude that $ab \geq 0$. This is true for all $p \in M^2$.

3. If $M^2$ is pseudo-umbilical then $a = b$ and by (3) we have that $G(p, e)$ takes maximum in $e_3$. To get further results we consider the normal curvature $R_{\beta \epsilon j}$ and scalar normal curvature $K_N$ [1]:

$$R_{\beta \epsilon j} = \sum_i (A_{3i k} A_{\beta \epsilon k} - A_{\beta \epsilon i} A_{3i k}),$$

$$K_N = \sum_{i, j, k} (A_{3i k} A_{\beta \epsilon j} - A_{\beta \epsilon i} A_{3i k})^2.$$

THEOREM 1. At points $p$ with $M$-index $M^2 \geq 2$, $M^2$ is pseudo-umbilical if and only if $G(p, e)$ has maximum in $e_3$; at points $p$ with $M$-index $M^2 = 1$, $M^2$ is pseudo-umbilical if and only if $G(p, e)$ has maximum in $e_3$ and $K_N = 0$; at points $p$ with $M$-index $M^2 = 0$, $M^2$ is pseudo-umbilical if and only if $M^2$ is totally umbilical.

PROOF. Suppose $M$-index $M^2 \geq 2$. Otsuki in [6] showed that $M$-index $M^2 \leq 2$. So $M$-index $M^2 = 2$. Since $m(A_3) = \frac{1}{2} (a + b) \neq 0$ and $m(A_r) = 0$ we have $\ker \bar{m} = \{ \sum_r \eta_r A_r, \eta_r \in R \}, \bar{e} \in \ker \bar{m}$ and $\psi_\beta(\bar{e}) = A_r$. Hence for at least one $r, e_r \neq 0$. But $G(p, e)$ takes maximum in $e_3$, we have $(a-b)c_r = 0$. Hence $a = b$ and $M^2$ is pseudo-umbilical. Next suppose $M$-index $M^2 = 1$, $G(p, e)$ has maximum in $e_3$ and $K_N = 0$. $A_r$ are given in (1). $\psi_\beta(\ker \bar{m}) = \{ \sum_r \eta_r A_r, \eta_r \in R \}$. If $\dim (\psi_\beta(\ker \bar{m})) = M$-index $M^2 = 1$ then there is $k$ so that $d_r = k\bar{e}$, for any $r$. We have then $A_3 A_r = A_r A_3$, and

$$K_N = 2 \sum_{i, j} (A_{3i k} A_{\beta \epsilon j} - A_{\beta \epsilon i} A_{3i k})^2.$$

$K_N = 0$ implies $A_3 A_r = A_r A_3$, that is $(a-b)d_r = 0$. If all $d_r = 0$ then at least one $c_r \neq 0$ because $M$-index $M^2 = 1$. Thus $G(p, e)$ having maximum at $e_3$ implies $a = b$. The inverse is clear. Finally suppose $M$-index $M^2 = 0$, then $c_r = d_r = 0$, i.e., $A_r = 0$. It is clear that $M^2$ is pseudo-umbilical, i.e., $a = b$, if and only if $M^2$ is totally umbilical.
4. In this section we assume that $M^2$ is a closed surface. For a symmetric matrix $A=(a_{ij})$ if we write $N(A)=\sum_{i,j} a_{ij}$, then we have $K_N=\sum_{a,\beta} N(A_a A_\beta - A_\beta A_a)$. Chen in [1] proved the following results: The Veronese surface is the only closed pseudo-umbilical surface in euclidean space with parallel normal curvature and scalar normal curvature $K_N\neq 0$, the 2-sphere and the Clifford torus are the only closed pseudo-umbilical surfaces in euclidean space with scalar normal curvature $K_N=0$ and scalar curvature $R\geq 0$. From these results we have the following two theorems.

**Theorem 2.** If $M^2$ is closed, pseudo-umbilical, $M$-index $M^2=1$ for any $p$ and the Gaussian curvature $G(p)\geq 0$ everywhere then $M^2$ is either a sphere or a Clifford torus.

**Proof.** If $M^2$ is pseudo-umbilical and $M$-index $M^2=1$ we have by Theorem 1 that $K_N=0$. The scalar curvature $R=2G\geq 0$ by assumption. By Chen’s result $M^2$ is either a sphere or a Clifford torus.

**Theorem 3.** If $M^2$ is closed, pseudo-umbilical, $M$-index $M^2=2$ for any $p$ and the Gaussian curvature $G(p)\geq (N-2)a^2/2N-3$ everywhere then $M^2$ is a Veronese surface.

**Proof.** $M$-index $M^2=2$ implies that $K_N\neq 0$. For the Laplacian of $A_{st}$ in the case of pseudo-umbilical $M^2$ we have the known equality [2]:

\[
\sum_{s,i,j} A_{sl}^i A_{sj} = 2a\Delta a + 2a^2 S - \sum_x S_x^2 - \sum_{x \neq \beta} N(A_x A_\beta - A_\beta A_x)
\]

where $S_x=\sum_{i,j} A_{xij}^i=N(A_x)$, $S=\sum_x S_x$. Since $M^2$ is pseudo-umbilical we have $A_3=aI$ and $\sum_{i,j} A_{3ij} \Delta A_{3ij}=2a\Delta a$. It is known also that [3] $N(A_a A_\beta - A_\beta A_a)\leq 2N(A_a)N(A_\beta)$. So we have $\sum_{r \neq t} N(A_r A_t - A_t A_r)\leq 2\sum_{r \neq t} N(A_r)N(A_t)=2\sum_{r \neq t} S_r S_t$. Let $S=\sum_r S_r$ and noticing that $2a^2 S_3=2a^2(2a^2)=4a^4=S^2$ we have from (7):

\[
\sum_{r,i,j} A_{ri}^i A_{rij} \geq 2a^2 S - \sum_r S_r^2 - 2 \sum_{r \neq t} S_r S_t
\]

\[
= 2a^2 S - \left( \sum_r S_r \right)^2 - 2 \sum_{r < t} S_r S_t.
\]

Let $\sigma_1$, $\sigma_2$ be such that $(N-1)\sigma_1=\sum_r S_r=S$, $(\frac{1}{2})(N-1)(N-2)\sigma_2=\sum_{r < t} S_r S_t$; it can easily be seen that [3]: $(N-1)^2(N-2)(\sigma_1^2-\sigma_2)=\sum_{r < t} (S_r - S_t)^2\geq 0$. Hence $\sigma_1^2 \geq \sigma_2$ or

\[
2 \sum_{r < t} S_r S_t \leq (N-2)S^2/(N-1).
\]
By (8) and (9) we have

\[(10) \sum_{r,i,j} A_{rij} \Delta A_{rij} \geq -(2N - 3)S^2/(N - 1) + 2a^2S.\]

The Gaussian curvature of $M^2$ is $G(p) = \sum_a \det(A_a) = a^2 - \sum_r (c_r^2 + d_r^2) = a^2 - (\frac{1}{2})S$. Therefore $S = 2(a^2 - G)$. By (10) we have

\[-\sum_{r,i,j} A_{rij} \Delta A_{rij} \leq 2S[(2N - 3)(a^2 - G)/(N - 1) - a^2].\]

Thus if $(2N - 3)(a^2 - G)/(N - 1) - a^2 \leq 0$ or $G \geq (N - 2)a^2/(2N - 3)$ then $\sum_{r,i,j} A_{rij} \Delta A_{rij} \geq 0$. Now from the equality

\[\frac{1}{2} \Delta \sum_{r,i,j} (A_{rij})^2 = \sum_{r,i,j,k} (A_{rjik})^2 + \sum_{r,i,j} A_{rij} \Delta A_{rij},\]

where $A_{rjik} = A_{rj,k}$ (covariant derivative), we have that if $G \geq (N - 2)a^2/(2N - 3)$ then $\sum_{r,i,j,k} (A_{rjik})^2 = 0$. By (11) it implies $A_{rij,k} = 0$. By (6) we have that $R_{jkl}^i$ (noticing that $R_{kjl}^i \equiv 0$) is parallel and that $K_N$ is constant and $K_N \neq 0$. By Chen’s result we conclude that $M^2$ is a Veronese surface.

**Theorem 4.** If $M^2$ is closed, $G(p, e)$ takes maximum in $e_3$ and $M$-index $M^2 = 0$ everywhere then $M^2$ is embedded as a convex surface in an $E^3$.

**Proof.** $M$-index $M^2 = 0$ implies that $c_r = d_r = 0$. Then we may write $G(p, e) = ab \cos^2 \theta_3$, $0 \leq \theta_3 \leq \pi$. By the lemma $ab \geq 0$. The Gaussian curvature $G(p)$ in this case is $G(p) = \sum_a \det(A_a) = ab$. So we have $G(p) \geq G(p, e)$ for all $e$. By the Gauss-Bonnet formula $\int_{M^2} G(p) \ dV = \int_{M^2} ab \ dV = 4\pi(1 - g)$. On the other hand the total curvature $K^*(p)$ of $M^2$ at $p$ is

\[K^*(p) = \int_{S^{N-1}} |G(p, e)| \ d\sigma_{N-1} = \int_{S^{N-1}} ab \cos^2 \theta_3 \ d\sigma_{N-1} = G(p)c_{N+1}/2\pi\]

where $S^{N-1}$ is the unit sphere in $E^N$ and $c_{N+1}$ is the volume of the unit sphere in $E^{2+N}$.

By a result due to Chern-Lashof [5] we have $(1/c_{N+1}) \int_{M^2} K^*(p) \ dV \geq 2 + 2g$, the equality sign holds if and only if $M^2$ is embedded as a convex surface in an $E^3$. On the other hand

\[(c_{N+1}/2\pi)4\pi(1 - g) = (c_{N+1}/2\pi)\int_{M^2} G(p) \ dV \]

\[= \int_{M^2} K^*(p) \ dV \geq c_{N+1}(2 + 2g).\]

That is $1 - g \geq 1 + g$. Thus it is necessary that $g = 0$ and the equality sign holds. Hence $M^2$ is embedded as a convex surface in $E^3$. 

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REFERENCES


