SUMS OF DISTANCES BETWEEN POINTS
ON A SPHERE
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Abstract. An upper bound for the sum of the $\lambda$th powers of all
distances determined by $N$ points on a unit sphere is given for
$0 \leq \lambda \leq 1$.

Let $p_1, \ldots, p_N$ be points on the unit sphere $U^m$ of $E^m$, the $m$-dimensional
Euclidean space. Let

$$S = S(N, m, \lambda) = \max \sum_{i \neq j} |p_i - p_j|^\lambda$$

where the maximum is taken over all possible $p_1, \ldots, p_N$. The function
$S$ has been studied in [1]-[5]. Let $\sigma(U^m)$ denote the surface area of
$U^m$. Björck has shown in [3] that $S < c(m, \lambda)N^2$ where $c(m, \lambda) = \frac{1}{2}\sigma(U^m)^{-1} \int |p_0 - p|^\lambda \, d\sigma(p)$; here $\int \cdots d\sigma(p)$ denotes a surface integral over
$U^m$ and $p_0$ is an arbitrary point on $U^m$. For example, $c(3, 1) = \frac{3}{2}$. In [1]
Alexander used Haar integrals to prove

$$(1) \quad S(N, 3, 1) < \frac{3}{2}N^2 - \frac{1}{2}.$$ 

By the same technique we will show (see comment after Theorem 2) that
for $0 \leq \lambda \leq 1$,

$$(2) \quad S(N, m, \lambda) \leq c(m, \lambda)N^2 - \frac{1}{2}c(m, \lambda)$$

is true at least “half of the time”; more precisely, for any positive integer
$n$ the inequality (2) is true either for $N = n$ or for $N = n + 1$. Our proof is
based on a new generalization of the triangle inequality, Theorem 1, and
an imbedding theorem of I. J. Schoenberg [6] or [7, p. 527]. Note that
when $m = 3$ and $\lambda = 1$ our result is weaker than Alexander’s.

Lemma 1. Let $u \geq v + 2 \geq 3$ and let $H$ denote the hexagonal region
$0 \leq x \leq u, 0 \leq y \leq v, 1 \leq x + y \leq u + v - 1$. Then for $(x, y)$ in $H$ we have

$$g(x, y) = \frac{x(u - x) + y(v - y)}{x(v - y) + y(u - x)} \leq \frac{u - 1}{v}.$$
Proof. Let $s$ denote the substitution which takes $x$ and $y$ to $u-x$ and $v-y$ respectively. The function $g(x,y)$ is invariant under $s$, so it suffices to prove the inequality for the three critical points $(\frac{1}{4}(u+v), \frac{1}{4}(u+v))$, $(\frac{1}{4}(u+v), \frac{1}{4}(u+v))$, $(\frac{1}{4}(u+v), \frac{1}{4}(u+v))$, and the three boundary lines $x+y=1$, $y=0$ and $x=u$. The straightforward details are omitted.

Now let $S_1$ be a finite set of $u$ points $p_1, \ldots, p_u$ in $E^m$ and similarly let $S_2$ be a set of $v$ such points $q_1, \ldots, q_v$. Let $\sum_1 = \sum_{i<j} |p_i - p_j|$, $\sum_2 = \sum_{i<j} |q_i - q_j|$, and $\sum_{12} = \sum_{i,j} |p_i - q_j|$.

Theorem 1. Let $u \geq v$. Define

$$f(u,v) = 1, \quad u = v,$$

$$= (u-1)/v, \quad u > v.$$

Then

$$\sum_1 + \sum_2 \leq f(u,v) \sum_{12}.$$

Moreover the constant $f(u,v)$ is best possible.

Proof. For any two points $p_1, p_2 \in E^m$,

$$|p_1 - p_2| = a_m \int |p_1 \cdot p - p_2 \cdot p| \, d\sigma(p)$$

where $a_m$ depends only on the dimension $m$. Hence it suffices to prove (3) in the case where the $p_i$ and $q_j$ are real numbers. Let $J$ be any interval on the real axis which does not contain any point $p_i$ or $q_j$. Let $x$ be the number of $p_i$ to the left of $J$ and $y$ the number of $q_j$ to the left of $J$. It suffices to show $L \leq f(u,v)R$ where $L$ is the number of times $J$ is counted on the left of (3) and $R$ is the number of times it is counted on the right, i.e. that

$$x(u-x) + y(v-y) \leq f(u,v)(x(v-y) + y(u-x)).$$

For $v \leq u \leq v+1$ this follows from $(x-y)(u-v) \leq (x-y)^2$ and for $v+2 \leq u$ it follows from Lemma 1. Equality holds in (3) if (not only if) $p_2 = p_3 = \cdots = p_u = q_1 = \cdots = q_v$ and $p_1$ is arbitrary.

Theorem 2. For $0 \leq \lambda \leq 1$,

$$s(u,m,\lambda) + s(v,m,\lambda) \leq 2c(m,\lambda)uvf(u,v).$$

Proof. Fix $\lambda$. Given any $t$ points $p_1, \ldots, p_t$ in $E^m$, we can find an $m' \geq m$ and $t$ points $p'_1, \ldots, p'_t$ in $E^{m'}$ such that $|p'_i - p'_j| = |p_i - p_j|^2$ for $1 \leq i, j \leq t$. This result is due to I. J. Schoenberg [7, p. 527]. In the following $\tau_1, \tau_2, \tau_3$ shall denote elements of the special orthogonal group $SO(m)$ acting on $U^m$. The symbol $\int \cdots \, d\tau$ shall denote a Haar integral over that group; we normalize the measure so that $\int_{SO(m)} \, d\tau = 1$. Choose $p_1, \ldots, p_u$.
and \( q_1, \ldots, q_v \) on \( U^m \) so that

\[
S(u, m, \lambda) = \sum_{i<j} |p_i - p_j|^2 \quad \text{and} \quad S(v, m, \lambda) = \sum_{i<j} |q_i - q_j|^2.
\]

Define

\[
I = \int \sum_{i,j} |p_i - \tau q_j|^2 \, d\tau.
\]

Clearly

\[
I = uv \int |p - \tau_1 \cdot p|^2 \, d\tau_1 = uuv\sigma(U^m)^{-1} \int |p_0 - p|^2 \, d\sigma(p) = 2c(m, \lambda)uv
\]

On the other hand, for any \( p_0 \) on \( U^m' \),

\[
|p_i - \tau \cdot q_j|^2 = |p_i' - (\tau_1 q_j)'| = b_{m'} \int |(p_i' - (\tau_1 \cdot q_j)' \cdot \tau_2 p_0| \, d\tau_2
\]

where \( b_{m'} \) depends only on the dimension \( m' \). Hence by Theorem 1,

\[
I = \iint b_{m'} \sum_{i,j} |p_i' \cdot \tau_2 p_0 - (\tau_1 q_j)' \cdot \tau_2 p_0| \, d\tau_2 \, d\tau_1
\]

\[
\geq f^{-1} \iint \left( \sum_{i<j} b_{m'} |(p_i' - p_j')| \cdot \tau_2 p_0 \right) \, d\tau_2 \, d\tau_1
\]

\[
+ \sum_{i<j} b_{m'} |(\tau_1 q_i)' - (\tau_1 q_j)' \cdot \tau_2 p_0| \right) \, d\tau_2 \, d\tau_1
\]

\[
=f^{-1} \iint \left( \sum_{i<j} |p_i - p_j|^2 + \sum_{i<j} |q_i - q_j|^2 \right) \, d\tau_1.
\]

The result follows from (5), (6), and (7).

To obtain (2) set \( v = u - 1 \) in (4) and note that \( cu^2 - \frac{1}{2}c + c(u-1)^2 - \frac{1}{2}c = 2cu(u-1) \).

**REFERENCES**