Sums of Distances between Points on a Sphere

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Abstract. An upper bound for the sum of the $\lambda$th powers of all distances determined by $N$ points on a unit sphere is given for $0 \leq \lambda \leq 1$.

Let $p_1, \ldots, p_N$ be points on the unit sphere $U^m$ of $E^m$, the $m$-dimensional Euclidean space. Let

$$S = S(N, m, \lambda) = \max \sum_{i<j} |p_i - p_j|^{\lambda}$$

where the maximum is taken over all possible $p_1, \ldots, p_N$. The function $S$ has been studied in [1]-[5]. Let $\sigma(U^m)$ denote the surface area of $U^m$. Björck has shown in [3] that $S \leq c(m, \lambda)N^2$ where $c(m, \lambda) = \frac{1}{2}\sigma(U^m)^{-1} \int |p_0 - p|^{\lambda} \, d\sigma(p)$; here $\int \cdots \, d\sigma(p)$ denotes a surface integral over $U^m$ and $p_0$ is an arbitrary point on $U^m$. For example, $c(3, 1) = \frac{2}{3}$. In [1] Alexander used Haar integrals to prove

$$S(N, 3, 1) \leq \frac{2}{3} N^2 - \frac{1}{2}.$$  \hspace{1cm} (1)

By the same technique we will show (see comment after Theorem 2) that for $0 \leq \lambda \leq 1$,

$$S(N, m, \lambda) \leq c(m, \lambda)N^2 - \frac{1}{2}c(m, \lambda)$$  \hspace{1cm} (2)

is true at least “half of the time”; more precisely, for any positive integer $n$ the inequality (2) is true either for $N=n$ or for $N=n+1$. Our proof is based on a new generalization of the triangle inequality, Theorem 1, and an imbedding theorem of I. J. Schoenberg [6] or [7, p. 527]. Note that when $m=3$ and $\lambda=1$ our result is weaker than Alexander’s.

Lemma 1. Let $u \geq v + 2 \geq 3$ and let $H$ denote the hexagonal region $0 \leq x \leq u$, $0 \leq y \leq v$, $1 \leq x+y \leq u+v-1$. Then for $(x, y)$ in $H$ we have

$$g(x, y) = \frac{x(u-x) + y(v-y)}{x(v-y) + y(u-x)} \leq \frac{u-1}{v}.$$
Proof. Let \( s \) denote the substitution which takes \( x \) and \( y \) to \( u-x \) and \( v-y \) respectively. The function \( g(x, y) \) is invariant under \( s \), so it suffices to prove the inequality for the three critical points \( (\frac{1}{2}(u+v), \frac{1}{2}(u+v)) \), \( (\frac{1}{2}(u+v), \frac{1}{2}(u+v)) \), \( (\frac{1}{2}u, \frac{1}{2}v) \), and the three boundary lines \( x+y=1 \), \( y=0 \) and \( x=u \). The straightforward details are omitted.

Now let \( S_1 \) be a finite set of \( u \) points \( p_1, \ldots, p_u \) in \( E^m \) and similarly let \( S_2 \) be a set of \( v \) such points \( q_1, \ldots, q_v \). Let \( \sum_1 = \sum_{i<j} |p_i-p_j| \), \( \sum_2 = \sum_{i<j} |q_i-q_j| \), and \( \sum_{12} = \sum_{i,j} |p_i-q_j| \).

**Theorem 1.** Let \( u \geq v \). Define
\[
f(u, v) = \begin{cases} 1, & u = v, \\ (u-1)/v, & u > v. \end{cases}
\]
Then
\[
\sum_1 + \sum_2 \leq f(u, v) \sum_{12}.
\]
Moreover the constant \( f(u, v) \) is best possible.

**Proof.** For any two points \( p_1, p_2 \in E^m \),
\[
|p_1 - p_2| = a_m \int |p_1 \cdot p - p_2 \cdot p| \ d\sigma(p)
\]
where \( a_m \) depends only on the dimension \( m \). Hence it suffices to prove (3) in the case where the \( p_i \) and \( q_j \) are real numbers. Let \( J \) be any interval on the real axis which does not contain any point \( p_i \) or \( q_j \). Let \( x \) be the number of \( p_i \) to the left of \( J \) and \( y \) the number of \( q_j \) to the left of \( J \). It suffices to show
\[
L \leq f(u, v) R \text{ where } L \text{ is the number of times } J \text{ is counted on the left of (3)}
\]
and \( R \) is the number of times it is counted on the right, i.e. that
\[
x(u-x) + y(v-y) \leq f(u, v) (x(v-y) + y(u-x)).
\]
For \( v \leq u \leq v+1 \) this follows from \( (x-y)(u-v) \leq (x-y)^2 \) and for \( v+2 \leq u \) it follows from Lemma 1. Equality holds in (3) if (not only if) \( p_2 = p_3 = \ldots = p_u = q_1 = \ldots = q_v \) and \( p_1 \) is arbitrary.

**Theorem 2.** For \( 0 \leq \lambda \leq 1 \),
\[
S(u, m, \lambda) + S(v, m, \lambda) \leq 2c(m, \lambda) uvf(u, v).
\]

**Proof.** Fix \( \lambda \). Given any \( t \) points \( p_1, \ldots, p_t \) in \( E^m \), we can find an \( m' \geq m \) and \( t \) points \( p'_1, \ldots, p'_t \) in \( E^{m'} \) such that \( |p_i - p'_j| = |p_i - p_j|^2 \) for \( 1 \leq i, j \leq t \). This result is due to I. J. Schoenberg [7, p. 527]. In the following \( \tau, \tau_1, \tau_2 \) shall denote elements of the special orthogonal group \( SO(m) \) acting on \( U^m \). The symbol \( \frac{1}{2} \cdots d\tau \) shall denote a Haar integral over that group; we normalize the measure so that \( \int_{SO(m)} d\tau = 1 \). Choose \( p_1, \ldots, p_u \).
and \( q_1, \ldots, q_v \) on \( U^m \) so that

\[
S(u, m, \lambda) = \sum_{i < j} |p_i - p_j|^2 \quad \text{and} \quad S(v, m, \lambda) = \sum_{i < j} |q_i - q_j|^2.
\]

Define

\[
I = \int \sum_{i, j} |p_i - \tau_1 q_j|^2 \, d\tau_1.
\]

Clearly

\[
I = uv \int |p - \tau_1 \cdot p|^2 \, d\tau_1 = uvo(U^m)^{-1} \int |p_0 - p|^2 \, d\sigma(p) = 2c(m, \lambda)uv
\]

On the other hand, for any \( p_0 \) on \( U^m' \),

\[
|p_i - \tau \cdot q_i|^2 = |p_i' - (\tau_1 q_i)'| = b_m' \int |(p_i' - (\tau_1 \cdot q_i))' \cdot \tau_2 p_0| \, d\tau_2
\]

where \( b_m' \) depends only on the dimension \( m' \). Hence by Theorem 1,

\[
I = \iint b_m' \sum_{i, j} |p_i' \cdot \tau_2 p_0 - (\tau_1 q_i)' \cdot \tau_2 p_0| \, d\tau_2 \, d\tau_1
\]

\[
\geq f^{-1} \iint \left( \sum_{i < j} b_m' |(p_i' - p_j') \cdot \tau_2 p_0| \right. \]

\[
\left. + \sum_{i < j} b_m' |(\tau_1 q_i)' - (\tau_1 q_j)' \cdot \tau_2 p_0| \right) \, d\tau_2 \, d\tau_1
\]

\[
= f^{-1} \int \left( \sum_{i < j} |p_i - p_j|^2 + \sum_{i < j} |q_i - q_j|^2 \right) \, d\tau_1.
\]

The result follows from (5), (6), and (7).

To obtain (2) set \( v = u - 1 \) in (4) and note that \( cu^2 - \frac{1}{2} c + c(u - 1)^2 - \frac{1}{2} c = 2cu(u - 1) \).

\section*{References}

1. R. Alexander, On the set of distances determined by \( n \) points on the 2-sphere (in preparation).