FILTER CHARACTERIZATIONS OF C- AND C*-EMBEDDINGS

JOHN WILLIAM GREEN

Abstract. A filter F on a space S is completely regular if the complement of each set in F is completely separated from some set in F. A characterization of the Stone-Cech compactification due to Alexandroff is used to establish the following theorem. Suppose K is a subspace of a Tychonoff space S. K is C*-embedded in S if and only if the trace on K of every maximal completely regular filter on S intersecting K is maximal completely regular on K. A similar characterization of the C-embedded subsets of a Tychonoff space is obtained as are several related results.

A characterization of the Stone-Cech compactification $\beta S$ of a Tychonoff space S due essentially to Alexandroff [1] is used to characterize the C*-embedded subspaces of S. This result is used to obtain a second characterization of such subspaces as well as one of the C-embedded subspaces. A few related results are obtained.

Throughout this paper, K will refer to a subspace of a Tychonoff space S. The notion of a completely regular filter was introduced in [1] under the term "completely regular system" and referred to a certain type of what is now called a filtersubbase. The term used here, as well as the reduction to filters, apparently was introduced by Bourbaki. (See, for example, [4, Chapter IX, §1, exercises].) The characterization of $\beta S$ given below may be found, at least implicitly, in [1], [3], [4], [5], [7] and, particularly, [9]. In [8], in several other papers, completely regular filters are used for distinct, though related, purposes. The reader is assumed to be familiar with the results in [4], as well as Chapter 6 of [6]. The terminology is that of these two sources, for the most part.

A filter F on S is completely regular if for each $U$ in F, there exist $V$ in F and $\phi$ in $L(S)$ (=the set of all functions in $C(S)$ with range a subset of $[0, 1]$) such that $\phi$ is 0 on $V$ and 1 on $S - U$. It should be noted that every completely regular filter has as base an e-filter [6, problem 2L] and the filter (in the lattice of all subsets of S) generated by an e-filter is completely regular. If $Y$ is the topology of S and for each $U \subseteq S$, $U^* = U \cup \{F : F$ is a

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free maximal completely regular filter on \( S \) having \( U \) as an element\)}, then \( B = \{U^*: U \in Y\} \) is a base for a topology on \( S^* \) with respect to which \( S^* \) is (homeomorphic to) \( \beta\mathcal{S} \). If \( x \) is a point of a space \( T \), \( \text{Nbd}_T(x) \) is the neighborhood filter of \( x \) in the space \( T \). If \( F \) is a filter on \( S \), \( F \) is said to intersect \( K \) if each set in \( F \) intersects \( K \), and \( F_K \) and \( \text{Tr}_K(F) \) are used for the trace of \( F \) on \( K \). A filterbase \( G \) is coarser than a filterbase \( F \) (written \( G \subseteq F \) or \( F \preceq G \)) if each set in \( G \) contains a set in \( F \). If \( F \) and \( G \) are filters on a set \( T \), \( \text{sup}(F, G) = \{U \subseteq T: U \supseteq f \cap g \text{ for some } f \in F \text{ and } g \in G\} \) and is a filter on \( T \), provided each set in \( F \) intersects each set in \( G \).

**Lemma.** If \( F \) is a maximal completely regular filter on \( K \), there is a unique maximal completely regular filter on \( S \) coarser than \( F \). Furthermore, if \( F \) is any free completely regular filter on \( K \), there is a coarser completely regular filter on \( S \) whose trace on \( K \) is free.

**Proof.** Suppose \( F \) is a free completely regular filter on \( K \) (relative to the subspace topology). Let \( G = \{S - \text{Cl}_x f: f \in F\} \). \( G \) is an \( S \)-open cover of \( K \) no finite subcollection of which covers \( K \). For each \( x \in K \), let \( U_x \) denote some open set in \( G \) containing \( x \) and \( \Phi_x = \{\phi \in L(S): \phi(x) = 1 \text{ and } \phi(S - U_x) = 0\} \). For each finite collection \( H \) of ordered pairs \((x, \phi)\) such that \( x \in K \) and \( \phi \in \Phi_x \), let \( \phi_H(t) = \text{sup}\{\phi(t): (x, \phi) \in H\} \), for each \( t \in S \). \( \phi_H \in L(S) \) and if \( 0 < e < 1 \), then (1) \( \phi_H^0[0, e) \supseteq K \), for if \( (x, \phi) \in H \), then \( \phi(x) = 1 \); and (2) \( \phi_H^1[0, e] \cap K \neq \emptyset \), for otherwise, \( K \subseteq \phi_H^0[0, 1] \subseteq \phi_H^1[0, 1] \subseteq \cup \{\phi^{-1}(0, 1): (x, \phi) \in H\} \subseteq \cup \{U_x: (x, \phi) \in H\} \), contrary to the fact that no finite subcollection of \( G \) covers \( K \). It follows that the filter \( F' \) on \( S \) with base \( \{\phi_H^1[0, e): 0 < e < 1, H \text{ is a finite collection of ordered pairs } (x, \phi) \text{ such that } x \in K \text{ and } \phi \in \Phi_x\} \) is completely regular on \( S \). It will be shown that \( F' \preceq F \). Suppose \( f' \in F' \). For some \( H = \{(x_n, \phi_n): n \leq p, x_n \in K, \phi_n \in \Phi_{x_n}\} \) and \( 0 < e < 1 \), \( f' \supseteq \phi_H^1[0, e) \). For each \( n \), \( \phi_n \) is \( 1 \) at \( x_n \) and for some \( f_n \in F \), is \( 0 \) on \( S - (S - \text{Cl}_x f_n) = \text{Cl}_x f_n \supseteq f_n \). \( \bigcap_{n \leq p} f_n = f \in F \). Thus, \( \phi_n (f) = 0 \) for each \( n \leq p \), so \( \phi_H (f) = 0 \). \( f' \supseteq \phi_H^0[0, e) \). Therefore, \( F' \preceq F \).

Therefore, every free completely regular filter on \( K \) is finer than some (not necessarily free) completely regular filter on \( S \) whose trace on \( K \) is free. A simple application of Zorn's lemma establishes the existence of a filter \( F' \) maximal with respect to the property of being a completely regular filter on \( S \) coarser than \( F \). \( F' \) is a maximal completely regular filter on \( S \) if \( F \) is on \( K \). For suppose there is a completely regular filter \( G \) on \( S \) strictly finer than \( F' \). \( G \preceq F \). \( \text{sup}(G_x, F) \) does not exist (as a filter), for if it does, it is a completely regular filter on \( K \) strictly finer than the maximal completely regular filter \( F \) on \( K \). It follows that there exist \( g \in G \) and \( f \in F \) such that \( \text{Cl}_x g \cap \text{Cl}_y f = \emptyset \). There exist \( g_1 \) in \( G \) and \( \phi \) in \( L(S) \).
such that \( \phi(g) = 1 \) and \( \phi(S - g) = 0 \). Let \( F'' = \sup\{F', \{\phi^{-1}[0, e): 0 < e < 1\}\} \). \( F'' \) is a completely regular filter on \( S \) strictly finer than \( F' \) and coarser than \( F \). This is contrary to the definition of \( F' \). Therefore, \( F'' \) is a maximal completely regular filter on \( S \). It is easily established that \( F'' \) is unique. If \( F \) is a fixed maximal completely regular filter on \( K \), then for some point \( x \) of \( K \), \( F = \text{Nbd}_K(x) = \text{Tr}_K(\text{Nbd}_S(x)) \).

Theorem 1. In order that \( K \) be \( C^* \)-embedded in \( S \), it is necessary and sufficient that the trace on \( K \) of every maximal completely regular filter on \( S \) intersecting \( K \) be maximal completely regular on \( K \).

Proof. The condition is sufficient. For suppose \( F \) is maximal completely regular on \( K \) and \( F' \) is the unique maximal completely regular filter on \( S \) coarser than \( F \). It is easily seen that \( F = F_K \). Let \( K' = K \cup \{F: F \text{ is a maximal completely regular filter on } S \text{ and } F_K \text{ is free}\} \). If \( F \in K' - K \) and is fixed, \( F \) is the neighborhood system in \( S \) of some point of \( S - K \) with which it will be identified. If \( F \in K' - K \) and is free, then \( F \) is a point in \( \beta S - S \) and \( \{f^* : f \in F\} \) is a base for the neighborhood filter in \( \beta S \) of the point \( F \). It is easily established that \( K' = \text{Cl}_{\beta S} K \) and hence is compact. Let \( \phi: K' \to \beta K \) such that \( \phi(x) = x \) if \( x \in K \), \( \phi(x) = \text{Tr}_K(\text{Nbd}_S(x)) \) if \( x \in K' \cap (S - K) \) and \( \phi(x) = x_K \) if \( x \in K' - S \). It is established above that \( \phi \) is a bijection.

Suppose \( x \in K' \) and \( U \) is a \( \beta K \)-open set containing \( \phi(x) \).

Case 1. Suppose \( x \in K \). There exists an \( S \)-open set \( D \) containing \( x \) such that \( D^*(K) = D \cap K \cup \{F : F \text{ is a free maximal completely regular filter on } K \text{ having } D \cap K \text{ as an element}\} \subseteq U \), \( \phi(D^* \cap K) \subseteq D^*(K) \). For suppose \( t \in D^* \cap K' \). \( \phi(t) = t \in D^*(K) \). Suppose \( t \in (D - K) \cap K' \) and \( F = \text{Nbd}_S(t) \). \( \phi(t) = F_K \) and since \( F \in D^* \), \( F_K \in D^*(K) \). Suppose \( t \in D^* \cap (K' - S) \). \( \phi(t) = t_K \) and since \( D \subseteq t \), \( D \cap K \subseteq t_K \). Thus, \( \phi(D^* \cap K') \subseteq D^*(K) \subseteq U \) and \( x \in D^* \cap K' \).

Case 2. Suppose \( x \in (K' - K) \cap S \). Let \( F = \text{Nbd}_S(x) \), \( \phi(x) = F_K \). There exists \( f \) in \( F \) such that \( f^*(K) \subseteq U \), \( F_K \in f^*(K) \) and \( F \in f^* \). That \( \phi(f^* \cap K') \subseteq f^*(K) \subseteq U \) is established much as in Case 1.

Case 3. Suppose \( x \in K' - S \). \( \phi(x) = x_K \). There exists \( f \in x \) such that \( f^*(K) \subseteq U \). As in Case 2, \( \phi(f^* \cap K') \subseteq f^*(K) \subseteq U \). Therefore, \( \phi \) is continuous. A direct proof that \( \phi^{-1} \) is continuous is not as simple, but homeomorphism is already established without that. So, \( \beta K \subseteq \beta S \) and \( K \) is \( C^* \)-embedded in \( S \).

The condition is necessary. For in this case, \( \beta K \subseteq \beta S \). If \( F \) is a maximal completely regular filter on \( S \) fixed at a point \( x \) of \( K \), then \( F = \text{Nbd}_S(x) \) and \( F_K = \text{Nbd}_K(x) \), which is maximal completely regular on \( K \). Suppose \( F \) is a maximal completely regular filter on \( S \) intersecting \( K \) such that \( F_K \) is free. There is a maximal completely regular filter on \( K \) finer than the
completely regular filter $F_K$. Suppose there are two, $G_1$ and $G_2$. $G_1$ and $G_2$ converge to distinct points of $\beta K$. Hence, $F$ accumulates at two points of $\beta S$, which is impossible. Let $F'$ denote the unique maximal completely regular filter on $K$ finer than $F_K$. Suppose $F' \neq F_K$. Then there is a set $f'$ in $F'$, open in $K$, and containing no set in $F_K$. Thus, for every closed $g$ in $F$, $g \cap K - f'$ is a nonempty set closed in $K$. Since $\beta K$ is compact, $\bigcap \{ \text{Cl}_K g \cap K - f' : g = \text{Cl}_S g \in F \}$ contains a point, $P$, which is a $\beta S$-accumulation point of $F$ but not of $F'$. $F'$ converges in $\beta K$ to $F' \neq P$. It follows that $F$ accumulates at the two points $P$ and $F'$, which is impossible. Therefore, $F_K = F'$.

**Corollary.** If $K$ is a discrete subspace of $S$, then $K$ is $C^*$-embedded in $S$ if and only if the trace on $K$ of every maximal completely regular filter on $S$ intersecting $K$ is an ultrafilter on $K$.

**Theorem 2.** In order that $K$ be $C^*$-embedded in $S$, it is necessary and sufficient that every maximal completely regular filter on $K$ be the trace on $K$ of a maximal completely regular filter on $S$.

**Proof.** The condition is sufficient. For suppose $F$ is a maximal completely regular filter on $S$ intersecting $K$. $F_K$ is completely regular on $K$, so there exists a maximal completely regular filter $G$ on $S$ such that $G_K$ is finer than $F_K$ and is maximal completely regular. Since $G_K$ and $F_K$ are compatible, so are $F$ and $G$; and since $F$ and $G$ are maximal, $F = G$. Thus, $F_K$ is maximal completely regular and the stated result follows from Theorem 1.

The necessity of the condition follows easily from Theorem 1 and the lemma.

**Theorem 3.** If $K$ is $C^*$-embedded in $S$, the trace on $K$ of every $z$-ultrafilter on $S$ intersecting $K$ is a $z$-ultrafilter on $K$.

**Proof.** Suppose $J$ is a $z$-ultrafilter on $S$ intersecting $K$. Let $F$ denote the unique maximal completely regular filter on $S$ coarser than $J$. $F_K \leq J_K$ and by Theorem 1 is maximal completely regular on $K$. There is a unique $z$-ultrafilter $Q$ on $K$ finer than $F_K$. Suppose there exist $U$ in $J_K$ and $V$ in $Q$ such that $U \cap V = \emptyset$. Then there exists $\phi \in L(K)$ such that $\phi^{-1}(0) = U$ and $\phi^{-1}(1) = V$. $\phi$ has a continuous extension $\phi_1$ in $L(S)$. $\phi_1^{-1}[0, 1) \in F$ since each set in $F$ intersects $\phi_1^{-1}[0, 1)$. Thus, the subset $U$ of $\phi_1^{-1}(1)$ fails to intersect some set in $F_K$ and yet $J_K \supseteq F_K$. This is a contradiction. Thus, each set in $Q$ intersects each set in $J_K$. Since $Q$ is a $z$-ultrafilter on $K$, $Q \supseteq J_K$. Suppose $V \in Q$. There exists $\phi$ in $L(K)$ such that $\phi^{-1}(0) = V$. $\phi$ has a continuous extension $\phi_1$ in $L(S)$. Since $V$ intersects every set in $J_K$, $\phi_1^{-1}(0)$ intersects every set in $J$ and thus belongs to $J$. Hence, $\phi_1^{-1}(0) \cap K = V \in J_K$. It follows that $Q = J_K$. 

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The converse of the above theorem is false, even if the closure in \( S \) of every zero set in \( K \) is a zero set in \( S \). In this regard, Lemma 3 of [2] may be of interest, where the normal base is the collection of all zero sets.

**Example.** Let \( S = [0, 1] \), \( K = [0, 1) \) with the usual topologies. Obviously, \( K \) is not \( C^* \)-embedded in \( S \). The only \( z \)-ultrafilters on \( S \) intersecting \( K \) are those fixed at a point of \( K \). If \( Z \) is a zero set in \( K \), then \( \text{Cl}_S Z \) is a zero set in \( S \) since it is closed and \( S \) is metric.

**Theorem 4.** In order that \( K \) be \( C \)-embedded in \( S \), it is necessary and sufficient that every \( z \)-ultrafilter on \( K \) be the trace of a \( z \)-ultrafilter on \( S \).

**Proof.** The condition is necessary. For by Theorems 1 and 3, \( \beta K \subseteq \beta S \), and the trace on \( K \) of every \( z \)-ultrafilter on \( S \) intersecting \( K \) is a \( z \)-ultrafilter on \( K \). Suppose \( F \) is a \( z \)-ultrafilter on \( K \). Let \( G \) denote the unique maximal completely regular filter on \( S \) coarser than \( F \), so that \( G \) is the unique maximal completely regular filter on \( K \) coarser than \( F \). Let \( J \) denote the unique \( z \)-ultrafilter on \( S \) finer than \( G \). Suppose some set \( U \) in \( J \) does not intersect \( K \). Since \( K \) is \( C \)-embedded in \( S \), there exists \( g \) in \( L(S) \) such that \( g^{-1}(0) \subseteq K \) and \( g^{-1}(1) \supseteq U \). For each \( e \) in \( (0, 1) \), \( g^{-1}[0, e) \in \mathcal{G} \) and hence, \( J \supseteq G \). This is a contradiction. Thus, \( J \) intersects \( K \) and \( J_K \) is a \( z \)-ultrafilter on \( K \). Since \( J \supseteq G \), \( J_K \supseteq G_K \). There is only one \( z \)-ultrafilter on \( K \) finer than \( G_K \). Hence, \( J_K = F \).

The condition is sufficient. It will first be shown that \( K \) is \( C^* \)-embedded in \( S \). It follows easily from the hypothesis that the trace on \( K \) of every \( z \)-ultrafilter on \( S \) intersecting \( K \) is a \( z \)-ultrafilter on \( K \). Suppose \( F \) is a \( z \)-ultrafilter on \( S \) finer than \( K \). Let \( G \) denote the unique maximal completely regular filter on \( S \) coarser than \( F \). Let \( J \) denote the unique \( z \)-ultrafilter on \( S \) finer than \( G \). Suppose some set \( U \) in \( J \) does not intersect \( K \). Since \( K \) is \( C \)-embedded in \( S \), there exists \( g \) in \( L(S) \) such that \( g^{-1}(0) \subseteq K \) and \( g^{-1}(1) \supseteq U \). For each \( e \) in \( (0, 1) \), \( g^{-1}[0, e) \in \mathcal{G} \) and hence, \( J \supseteq G \). This is a contradiction. Thus, \( J \) intersects \( K \) and \( J_K \) is a \( z \)-ultrafilter on \( K \). Since \( J \supseteq G \), \( J_K \supseteq G_K \). There is only one \( z \)-ultrafilter on \( K \) finer than \( G_K \). Hence, \( J_K = F \).
Suppose $K$ is not $C$-embedded in $S$. From Theorem 1.18 of [6], it follows that there is a zero set $Z$ in $S$ not intersecting $K$ such that if $g \in C^*(S)$ and $g^{-1}(0) = Z$, then for each $e > 0$, $g^{-1}[0, e] \cap K \neq \emptyset$. Let $F_1 = \{g^{-1}[0, e] \cap K : 0 < e, g \in C^*(S) \text{ and } g^{-1}(0) = Z\}$. $F_1$ is a base for a $z$-filter on $K$. Hence, there is a $z$-ultrafilter $F$ on $K$ finer than $F_1$. $F$ is the trace on $K$ of some $z$-ultrafilter $J$ on $S$, by hypothesis. $Z \notin J$, since $Z \cap K = \emptyset$, so there exists $V \in J$ such that $V \cap Z = \emptyset$. Since $Z$ and $V$ are zero sets in $S$, there exists $g$ in $L(S)$ such that $g^{-1}(0) = Z$ and $g^{-1}(1) = V$. But if $0 < e < 1$, $g^{-1}[0, e] \cap K \in F \subseteq J_K$ and thus, $g^{-1}[0, e] \cap V \neq \emptyset$. This is a contradiction. Therefore, $K$ is $C$-embedded in $S$.

A minor modification of the argument in the last paragraph above establishes the following.

**Theorem 5.** If $K$ is $C^*$-embedded in $S$ and every $z$-ultrafilter on $K$ is finer than some $z$-ultrafilter on $S$, then $K$ is $C$-embedded in $S$.

The following summary of Theorems 2 and 4 was suggested by the referee. It should be noted, however, that while the trace of a completely regular filter on $S$ on an arbitrary subset $K$ is completely regular on $K$, the same is not true of $e$-filters without some restriction on $K$.

**Theorem 6.** $K$ is $C$- [$C^*$-] embedded in $S$ if and only if every $z$- [$e$-] ultrafilter on $K$ is the trace of a $z$- [$e$-] ultrafilter on $S$.

**Theorem 7.** If $K$ is countable, then $K$ is $C$-embedded in $S$ if and only if $K$ is completely separated from every zero set in $S$ not intersecting $K$.

**Proof.** Suppose $K$ is completely separated from every zero set in $S$ not intersecting $K$. It follows from 3B.1 of [6] that $K$ is closed and completely separated from every closed set not intersecting $S$. Suppose $K_1$ and $K_2$ are subsets of $K$ completely separated in $K$. There exists $\phi$ in $L(K)$ such that $\phi(K_1) = 0$ and $\phi(K_2) = 1$. Since $K$ is countable, there exists $0 < r < 1$ such that $\phi^{-1}(r) \cap K = \emptyset$. It follows that $K_1 = K \cap \phi^{-1}(0, r]$ and $K_2 = K \cap \phi^{-1}[r, 1)$ are completely separated in $K$, contain $K_1$ and $K_2$, respectively, and $K = K_1 \cup K_2$.

Every closed subset of $K$ is the intersection of $K$ and a zero in $S$. For suppose $H$ is a closed subset of $K$. For each $x$ in $K - H$, there is a zero set $Z_x$ in $S$ containing $H$ but not containing $x$. $\bigcap Z_x$ is the intersection of countably many zero sets in $S$ and thus is a zero set whose intersection with $K$ is $H$.

Thus, there exist zero sets $Z_1$ and $Z_2$ in $S$ such that $Z_1 \cap K = K_1$ and $Z_2 \cap K = K_2$. $Z_1 \cap Z_2$ is a zero set not intersecting $K$ and so, by hypothesis, there is a zero set $Z$ in $S$ containing $K$ and not intersecting $Z_1 \cap Z_2$. $Z \cap Z_1$ and $Z \cap Z_2$ are mutually exclusive zero sets in $S$ containing $K_1$ and $K_2$. 

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$K_2$, respectively. Hence, each two sets completely separated in $K$ are completely separated in $S$. By Urysohn’s extension theorem, $K$ is $C^*$-embedded in $S$. It follows from Theorem 1.18 of [6] that $K$ is $C$-embedded in $S$. That the converse is true is obvious.

Thus, statements 1 and 3 of problem 3L.4 of [6] remain equivalent even if the requirement that $D$ be discrete is omitted.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73069