TOROIDAL ARCS ARE CELLULAR

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ABSTRACT. We prove that a toroidal, cell-like, locally connected continuum is cellular.

1. Introduction. An arc may be a decreasing intersection of cubes-with-two-handles and still not be cellular, or even toroidal. An arc formed by joining two Fox-Artin arcs [4] at their tame ends serves as an example.

However, it follows directly from the theorem below that all toroidal arcs are cellular. The theorem generalizes Daverman's result [3] concerning toroidal 3-cells. It was suggested to me as a problem by D. R. McMillan.

2. Definitions. A continuum X in a 3-manifold $M^3$ is cellular if $X = \bigcap_{i=1}^{\infty} X_i$ where $X_{i+1} \subseteq \text{Int } X_i$ for each i, and each $X_i$ is a 3-cell in $M^3$.

A continuum X in $M^3$ is toroidal if $X = \bigcap_{i=1}^{\infty} X_i$ where $X_{i+1} \subseteq \text{Int } X_i$ for each i, and each $X_i$ is a solid torus in $M^3$.

A continuum X in $M^3$ is cell-like if for any neighborhood U of X in $M^3$, there is a neighborhood V of X in $M^3$ such that V is homotopically trivial in U.

3. Theorem. A toroidal, cell-like, locally connected continuum is cellular.

Proof. Assume the continuum X is toroidal and cell-like but not cellular. Then $X = \bigcap_{i=0}^{\infty} T_i$ where for each i, $T_i$ is a solid torus and $T_{i+1} \subseteq \text{Int } T_i$. Since X is cell-like, we may assume that for each i the winding number of $T_{i+1}$ in $T_i$ is zero (that is, $T_{i+1}$ is homotopically trivial in $T_i$). Since X is not cellular, we may assume that for each i the wrapping number of $T_{i+1}$ in $T_i$ is not zero (that is, each meridional disk of $T_i$ intersects $T_{i+1}$). Let $D$, $E$, $F$, and $G$ be four disjoint polyhedral meridional disks of $T_0$ with $F$ and $G$ in different components of $T_0 - (D \cup E)$. We may assume that for each i, $T_i$ is polyhedral and Bd $T_i$ is in general position with $\Delta = D \cup E \cup F \cup G$. $\Delta \cap \text{Bd } T_i$ is a finite collection of trivial and meridional (with respect to Bd $T_i$) simple closed curves, because the wrapping number of $T_i$ in $T_0$ is not zero [2, Theorem 1].
If $\Delta \cap \partial T_1$ has any trivial simple closed curves on $\partial T_1$, choose $J$ to be an innermost trivial curve on $\partial T_1$. $J$ bounds a disk $D'$ on $\partial T_1$ whose interior misses $\Delta$.

$J$ lies in one of the four disks $D$, $E$, $F$, or $G$, say $D$, and bounds a disk $D''$ there. Replace $D$ by $(D - D'') \cup D'$ and then push $D'$ slightly off $\partial T_1$ to the appropriate side. We can remove all trivial curves in this way.

We have four new disjoint meridional disks $D_1$, $E_1$, $F_1$, and $G_1$.

Similarly change $(D_1, E_1, F_1, G_1)$ to $(D_2, E_2, F_2, G_2)$ so that if $\Delta_2 = D_2 \cup E_2 \cup F_2 \cup G_2$ then $\Delta_2 \cap \partial T_2$ contains no trivial simple closed curves of $\partial T_2$ for $j \leq 2$.

Continue this process to get a sequence $(D_1, E_1, F_1, G_1), \ldots, (D_n, E_n, F_n, G_n), \ldots$ where for each $n$, if $\Delta_n = D_n \cup E_n \cup F_n \cup G_n$ then $(\partial T_n) \cap \Delta_n$ contains no trivial curves of $\partial T_n$ for $J \leq n$.

We use this construction to prove the following lemma.

**Lemma.** There are infinitely many components of $X - \Delta$ each of whose closures intersect two of the disks $D$, $E$, $F$, and $G$.

**Proof.** It is clearly enough to show that given $n > 0$ there are at least $n$ such components. It is also enough to show that given $n > 0$, $\exists m > 0$ such that there are at least $n$ components of $X - \Delta_m$ whose closures intersect two of $D_m$, $E_m$, $F_m$, and $G_m$. In fact, each component of $X - \Delta_m$ whose closure intersects, say, $D_m$ and $F_m$ contains a component of $X - \Delta_m$ whose closure intersects $D$ and $F$. To see this let $C$ be a component of $X - \Delta_m$ such that $C$ intersects $D_m$ and $F_m$. Then $C$ intersects $D$ and $F$ since $\Delta_m \cap X \subseteq \Delta \cap X$. By a theorem of elementary topology $C$ contains an irreducible continuum $C'$ from $D$ to $F$ and $C' - (D \cup F)$ is connected. The component of $X - \Delta$ containing $C' - (D \cup F)$ lies in $C$ and its closure intersects $D$ and $F$.

Now, fixing $N > 0$, consider a homotopy core $J$ of $T_N$ lying in $\partial T_N$. Also take $J$ so that it intersects each curve of $(\partial T_N) \cap \Delta_N$ just once. $J$ must contain a subarc from $D_N$ to $E_N$. Without loss of generality assume this arc lies in the $F_N$ half of $T_0 - (D_N \cup E_N)$. The existence of this arc assures the existence of a cylinder in $\partial T_N$ with one end in $D_N$ and one in $E_N$. A **spanning cylinder** is an annulus with interior in the $F_N$ half of $T_0 - (D_N \cup E_N)$ and with one boundary component in $D_N$ and one in $E_N$. Spanning cylinders are defined only for the integer $N$. A spanning cylinder $A$ is said to be **inside** a spanning cylinder $B$ if $D_N \cup E_N \cup B$ separates $\text{Int } A$ from $\partial T_0$. **Inside B** is the bounded closed component of $T_0 - (D_N \cup E_N \cup B)$. Choose an outermost spanning cylinder $C_N, 1 \subseteq \partial T_N$. $C_N, 1$ must lie inside an outermost spanning cylinder $C_N - 1, 1 \subseteq \partial T_N - 1$. The following linking argument assures the existence of another outermost spanning cylinder $C_N, 2 \subseteq \partial T_N$ inside $C_N - 1, 1$.
Suppose $C_{N,1}$ is the unique outermost spanning cylinder of $\text{Bd} \ T_N$ inside $C_{N-1,1}$. Let $J_1=C_{N,1} \cap D_N$ and $J_2=C_{N-1,1} \cap D_N$. Pull $J$ slightly off $C_{N,1}$ to the inside of $T_N$. Then we still have $J \subseteq T_N$. $J_1$ is a meridian of $\text{Bd} \ T_N$, so the disk in $D_N$ bounded by $J_1$ contains an odd number of points of $J$. $J_2$ is a meridian of $\text{Bd} \ T_{N-1}$, so since the winding number of $T_N$ in $T_{N-1}$ is zero, the disk in $D_N$ bounded by $J_2$ contains an even number of points of $J$. Then the annulus in $D_N$ bounded by $J_1$ and $J_2$ contains an odd number of points of $J$. However, by the uniqueness assumption, each point of $J$ in the annulus is the endpoint of subarc of $J$ which runs from $D_N$ into the $F_N$ half of $T_N-(D_N \cup E_N)$ and back to $D_N$ again. Therefore the number of points of $J$ in the annulus is even, a contradiction.

Now $C_{N-1,1}$ is inside an outermost spanning cylinder $C_{N-2,1}$ of $T_{N-2}$. The linking argument gives us another outermost spanning cylinder $C_{N-1,2}$ of $\text{Bd} \ T_{N-1}$ inside $C_{N-2,1}$. After $N$ applications of the linking argument we have spanning cylinders $C_{i,j}$, $1 \leq i \leq N$ and $j \leq 2$, where each $C_{i,j} \subseteq \text{Bd} \ T_i$ and is an outermost such cylinder. In addition $C_{i,1}$ is inside $C_{i-1,1}$ for $1 < i \leq N$. Thus

$$\text{inside} \ C_{i,2} \cap \text{inside} \ C_{j,2} = \emptyset \quad \text{for} \ i \neq j.$$

The lemma will be proved if we can find a component of $X-\Delta_N$ inside each $C_{i,2}$ whose closure intersects $F_N$ and one of $D_N$ or $E_N$. If we knew $X \cap F_N \cap \text{inside} \ C_{i,2} \neq \emptyset$, then we could find an irreducible continuum $C'$ in $X$ from $F_N \cap \text{inside} \ C_{i,2}$ to $(D_N \cup E_N) \cap \text{inside} \ C_{i,2}$. The component of $X-\Delta_N$ containing $C'-\Delta_N$ would be the desired component. Then we need only show $X \cap F_N \cap \text{inside} \ C_{i,2} \neq \emptyset$. Take an innermost (in $F_N$) simple closed curve $J$ of $F_N \cap C_{i,2}$. $J$ is meridional on $T_i$ and bounds a disk $H$ in $F_N$ which intersects $X$ only if $F_N$ does inside $C_{i,2}$. Choose an innermost (in $H$) curve $J'$ of $H \cap \text{Bd} \ T_i$. $J'$ bounds a disk $H'$ in $H$ and inside $C_{i,2}$. $H'$ is a meridional disk of $T_i$ so $H' \cap X \neq \emptyset$ and therefore $X \cap F_N \cap \text{inside} \ C_{i,2} \neq \emptyset$ and the lemma is proved.

The theorem now follows easily from the lemma. There are infinitely many components of $X-\Delta$ whose closures intersect two of these disks, say $D$ and $F$. It follows easily that $X$ is not locally connected.

4. Corollary. Suppose $X$ is a cell-like, locally connected continuum in $S^3$ and $X=\bigcap_{i=0}^\infty X_i$ where $X_{i+1} \subseteq \text{Int} \ X_i$ and each $X_i$ is a 3-manifold bounded by a torus (a solid torus or a cube-with-a-knotted-hole). Then $X$ is cellular.

Proof. Assume $X$ satisfies the hypotheses of the corollary but $X$ is not cellular. By the theorem $X$ is not toroidal, so we may assume each $X_i$ is a cube-with-a-knotted-hole [1]. Since $X$ is cell-like we may assume $X_{i+1}$ is homotopically trivial in $X_i$ for each $i$. Since $X$ is not cellular we
may assume no $X_i$ contains a cell with $X$ in its interior. Also take each $X_i$ to be polyhedral.

Now let $T_i = S^3 - \text{Int } X_i$ for each $i$. Then $T_i \subseteq \text{Int } T_{i+1}$ and each $T_i$ is a knotted solid torus (this is the definition of a cube-with-a-knotted-hole).

**Property 1.** For each $i$, there is no meridional disk of $T_{i+1}$ missing $T_i$.
For if there were such a polyhedral disk $D$, a closed regular neighborhood $N(X_i + X \cup D)$ would be a cell in $X_i$ containing $X$ in its interior.

**Property 2.** $T_i$ is homotopically trivial in $T_{i+1}$, or $T_i \sim 0$ in $T_{i+1}$. Since $X_{i+1} \sim 0$ in $X_i$, if $J$ is a meridional simple closed curve of $T_{i+1}$, then $J \sim 0$ in $S^3 - T_i$. So the winding number of $T_i$ in $T_{i+1}$ is zero, so $T_i \sim 0$ in $T_{i+1}$.

However Kister and McMillan [5] showed that the union of an ascending sequence of knotted solid tori with the Properties 1 and 2 cannot be imbedded in $S^3$. The corollary is thus proved by contradiction.

**References**


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